

A Nuclear Weyl Algebra

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Abstract

A bilinear form on a possibly graded vector space V defines a graded Poisson structure on its graded symmetric algebra together with a star product quantizing it. This gives a model for the Weyl algebra in an algebraic framework, only requiring a field of characteristic zero. When passing to \mathbb{R} or \mathbb{C} one wants to add more: the convergence of the star product should be controlled for a large completion of the symmetric algebra. Assuming that the underlying vector space carries a locally convex topology and the bilinear form is continuous, we establish a locally convex topology on the Weyl algebra such that the star product becomes continuous. We show that the completion contains many interesting functions like exponentials. The star product is shown to converge absolutely and provides an entire deformation. We show that the completion has an absolute Schauder basis whenever V has an absolute Schauder basis. Moreover, the Weyl algebra is nuclear iff V is nuclear. We discuss functoriality, translational symmetries, and equivalences of the construction. As an example, we show how the Peierls bracket in classical field theory on a globally hyperbolic spacetime can be used to obtain a local net of Weyl algebras.

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1 Introduction

The Weyl algebra as the mathematical habitat of the canonical commutation relations has many incarnations and variants: in a purely algebraic definition it is the universal unital associative algebra generated by a vector space V subject to the commutation relations $vw - wv = \Lambda(v, w)\mathbb{1}$ where Λ is a symplectic form on V and $v, w \in V$. For $V = \mathbb{k}^2$ with basis q, p and the standard symplectic form, the canonical commutation relations take the familiar form

$$qp - pq = \mathbb{1}. \quad (1.1)$$

Typically, some scalar prefactor in front of $\mathbb{1}$ is incorporated. When working over the complex numbers, a C^* -algebraic version of the canonical commutation relations is defined by formally exponentiating the generators from V and using the resulting commutation relations from (1.1) with $i\hbar$ in front of $\mathbb{1}$ for the exponentials. It results in a universal C^* -algebra generated by the exponentials subject to the commutation relations. This version of the Weyl algebra is most common in the axiomatic approaches to quantum field theory and quantum mechanics. More recently, an alternative construction of a Weyl algebra in a C^* -algebraic framework has been proposed and studied in [9] based on resolvents instead of exponentials. In [5] the C^* -algebraic Weyl algebra was shown to be a strict deformation quantization of a certain Poisson algebra which consists of certain “bounded” elements in contrast to the “unbounded” generators from V itself, ultimately leading to a continuous field of C^* -algebras.

The two principle versions of the Weyl algebra differ very much in their behaviour. While the C^* -algebraic formulation has a strong analytical structure, the algebraic version based on the canonical commutation relations does not allow for an obvious topology: in fact, it can easily be proven that any submultiplicative seminorm on the Weyl algebra generated by q and p with commutation relations (1.1) necessarily vanishes. In particular, there will be no structure of a normed algebra possible.

It is now our main objective of this paper to fill this gap and provide a reasonable topology for the algebraic Weyl algebra making the product continuous. Starting point will be a general locally convex topology on V , where we also allow for a graded vector space and a graded version of the canonical commutation relations, i.e. we treat the Weyl algebra and the Clifford algebra on the same footing. The algebraic version of the Weyl algebra will be realised by means of a deformation quantization [2] of the symmetric algebra $S^\bullet(V)$ encoded in a star product. The idea is to treat the Poisson bracket arising from the bilinear form Λ as a *constant* Poisson bracket and use the Weyl-Moyal star product quantizing it. From a deformation quantization point of view this is a very trivial situation, though we of course allow for an infinite-dimensional vector space V , see [24] for a gentle introduction to deformation quantization. The bilinear form Λ will not be required to be antisymmetric or non-degenerate. However, we need some analytical properties. For convenience, we require Λ to be *continuous*, a quite strong assumption in infinite dimensions. Nevertheless, many interesting examples fulfill this requirement. In particular, for a finite-dimensional space V this is always the case.

On the tensor algebra $T^\bullet(V)$ and hence on the symmetric algebra $S^\bullet(V)$ there is of course an abundance of locally convex topologies which all induce the projective topology on each $V^{\otimes n}$. The two extreme cases are the direct sum topology and the Cartesian product topology. The direct sum topology for the Weyl algebra with finitely many generators was used in [12] to study bivariate K -theory. In [6] a slightly coarser topology on $S^\bullet(\mathcal{S}(\mathbb{R}^d))$ than the direct sum topology was studied in the context of quantum field theories, where $\mathcal{S}(\mathbb{R}^d)$ is the usual Schwartz space. It turns out that this topology makes $S^\bullet(\mathcal{S}(\mathbb{R}^d))$ a topological algebra, too. However, for our purposes, this topology is still too fine. Interesting new phenomena are found in [13] for formal star products in the case the underlying locally convex space V is a Hilbert space. For the class of functions considered in this paper, the classification program shows much richer behaviour than in the well-known finite-dimensional case. However, the required Hilbert-Schmidt property will differ from the requirements we state in the sequel.

The first main result is that we can define a new locally convex topology on the tensor algebra $T^\bullet(V)$ and hence also on the symmetric algebra $S^\bullet(V)$, quite explicitly by means of seminorms controlling the growth of the coefficients $a_n \in S^n(V)$, in such a way that the star product is continuous. The completion of $S^\bullet(V)$ with respect to this locally convex topology will contain many interesting entire functions like exponentials of elements in V . It turns out that even more is true: the star product converges absolutely and provides an entire deformation in the sense of [23].

If the underlying vector space V has an absolute Schauder basis we prove that the corresponding Weyl algebra also has an absolute Schauder basis. The second main result is that the Weyl algebra is (strongly) nuclear whenever we started with a (strongly) nuclear V . This is of course a very desirable property and shows that the Weyl algebra enjoys some good properties. In the case where V is finite-dimensional, we have both for trivial reasons: an absolute Schauder basis of V and strong nuclearity. Thus in this case the corresponding Weyl algebra turns out to be a strongly nuclear algebra with an absolute Schauder basis. In fact, we can show even more: the underlying locally convex space is a particular Köthe space which can explicitly be described. This way, we reproduce the earlier results from [3, 4], thereby clarifying these more ad-hoc constructions.

Our construction depends functorially on the data V and Λ as well as on a parameter R which controls the coarseness of the topology. The particular value $R = 1$ seems to be distinguished for several reasons. We show that the topological dual V' acts on the Weyl algebra by translations. These automorphisms are even *inner* if the element in V' is in the image of the canonical map $V \rightarrow V'$ induced by the antisymmetric part of Λ : here we show that the exponential series of elements in V are contained in the Weyl algebra, provided $R \leq 1$. Since the Weyl algebra does not allow for a general holomorphic calculus, this is a nontrivial statement and puts heuristic formulas for the star-exponential on a solid ground. In particular, these exponentials are also the generators of the C^* -algebraic version of the Weyl algebra, showing that there is still a close relation. However, it does not seem to be easy to make the transition to the C^* -algebraic Weyl algebra more explicitly. Note that for a finite-dimensional even vector space V , this reproduces the results from [3, 4].

Finally, we apply our general construction to an example from (quantum) field theory. We consider a linear field equation on a globally hyperbolic spacetime manifold. The Green operators of the normally hyperbolic differential operator encoding the field equation define a Poisson bracket, the so-called Peierls bracket. We show the relevant continuity properties in order to apply the construction of the Weyl algebra to this particular Poisson bracket. It is shown that the resulting Poisson algebra and Weyl algebra relate to the canonical Poisson algebra and Weyl algebra on the initial data of the field equation. The result will be a local net of Poisson algebras or Weyl algebras obeying a version of the Haag-Kastler axioms including the time-slice axiom. On one hand this is a very particular case of the Peierls bracket discussed in [7], on the other hand, we provide a simple quantum theory with honestly converging star product in this situation thereby going beyond the formal star products as discussed in [14, 15]. It would be very interesting to see how the much more general (and non-constant) Poisson structures in [7] can be deformation quantized with a convergent star product.

To conclude this introduction we take the opportunity to point out some possible further questions arising with our approach to the Weyl algebra:

- In finite dimensions it is always possible to choose a compatible almost complex structure for a given symplectic Poisson structure. Such a choice gives a star product of Wick type where the symmetric part of Λ now consists of a suitable multiple of the compatible positive definite inner product. The Wick product enjoys the additional feature of being a *positive* deformation [10]. In particular, the evaluation functionals at the points of the dual will become positive linear functionals on the Wick algebra. In [3] the corresponding GNS construction was investigated in detail and yields the usual Bargmann-Fock space representation for the canonical commutation relations. The case of a Hilbert space of arbitrary dimension will be the natural generalization for this. In general, the existence of a compatible almost complex structure having good continuity

properties is far from being obvious. We will address these questions in a future project.

- Closely related will be the question what the states of the locally convex Weyl algebra will be in general. While this question might be quite hard to attack in full generality, the more particular case of the Weyl algebra arising from the Peierls bracket will be already very interesting: here one has certain candidates of so-called Hadamard states from (quantum) field theory. It would be interesting to see whether and how they can be matched with compatible almost complex structures and evaluations at points in the dual.
- In infinite dimensions there are important examples of bilinear forms which are not continuous but only separately continuous. It would be interesting to extend our analysis to this situation as well in such a way that one obtains a separately continuous star product.
- Finally, already in finite dimensions it will be very challenging to go beyond the geometrically trivial case of constant Poisson structures. One possible strategy is to use the completed nuclear Weyl algebra build on each tangent space of a symplectic manifold. This leads to a Weyl algebra bundle, now in our convergent setting. In a second step one should try to understand how the Fedosov construction [16] of a formal star product can be transferred to this convergent setting provided the curvature and its covariant derivatives of a suitably chosen symplectic connection satisfies certain (still to be found) bounds.

The paper is organised as follows: In Section 2 we fix our notation and recall some well-known algebraic facts on constant Poisson structures and their deformation quantizations. The next section contains the core results of the paper. We first construct several systems of seminorms on the tensor algebra and investigate the continuity properties of the tensor product with respect to them. The continuity of the Poisson bracket is then established but the continuity of the star product requires a suitable projective limit construction in addition. Nevertheless, the resulting systems of seminorms are still described explicitly. This way, we arrive at our definition of the Weyl algebra in Definition 3.13 and show that it yields a locally convex algebra in Theorem 3.19. We prove that the star product converges absolutely, provides an entire deformation, and enjoys good reality properties. In Section 4 we show two main results: first that if V has an absolute Schauder basis the Weyl algebra also has an absolute Schauder basis. Second, we prove in Theorem 4.10 that the Weyl algebra is (strongly) nuclear iff V is (strongly) nuclear. Section 5 is devoted to various symmetries and equivalences. We prove that the algebraic symmetries can be cast into the realm of the locally convex Weyl algebra, too, and yield a good functoriality of the construction. If the convergence parameter R is at most 1 then translations are shown to act by inner automorphisms. Moreover, we show that the isomorphism class of the Weyl algebra only depends on the antisymmetric part of the bilinear form Λ . Finally, we show in Proposition 5.14 that for a finite-dimensional vector space our construction reproduces the results from [3, 4]. The final and quite large Section 6 contains a first nontrivial example: the canonical and the covariant Poisson structures arising in non-interacting field theories on globally hyperbolic spacetimes. We recall the necessary preliminaries to define and compare the two Poisson structures in detail. The continuity properties of both allow to apply our general construction of the Weyl algebra, leading to a detailed description in Theorem 6.14. As a first application we show that both on the classical side as well as on the quantum side the construction leads to a local net of observables satisfying the time-slice axiom.

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2 Algebraic Preliminaries

In this section we collect some algebraic preliminaries on constant Poisson brackets and their deformation quantization to fix our conventions. In the following, let \mathbb{k} be a field of characteristic 0 and let V be a \mathbb{k} -vector space.

2.1 The Symmetric Algebra $S^\bullet(V)$ and Sign Conventions

In order to treat the symmetric and the Grassmann algebra on the same footing, we assume that $V = V_0 \oplus V_1$ is \mathbb{Z}_2 -graded. In many applications, V is even \mathbb{Z} -graded and the induced \mathbb{Z}_2 -grading is then given by the even and odd part of V . A vector $v \in V_0$ is called homogeneous of parity **0** while a vector in V_1 is called homogeneous of parity **1**. Occasionally, we denote the parity of the vector v with the same symbol $v \in \mathbb{Z}_2$ and we also shall refer to even and odd parity. In all what follows, we will make use of the *Koszul sign rule*, i.e. if two things with parities $a, b \in \mathbb{Z}_2$ are exchanged this gives an extra sign $(-1)^{ab}$. The homogeneous components of $v \in V$ will be denoted by $v = v_0 + v_1$.

In more detail, we will need to following signs for *symmetrization*. For homogeneous vectors $v_1, \dots, v_n \in V$ and a permutation $\sigma \in S_n$ one defines the sign

$$\text{sign}(v_1, \dots, v_n; \sigma) = \prod_{i < j} \frac{\sigma(i) + (-1)^{v_{\sigma(i)} v_{\sigma(j)}} \sigma(j)}{i + (-1)^{v_i v_j} j}. \quad (2.1)$$

Then $\text{sign}(v_1, \dots, v_n; \sigma) = 1$ if all the v_1, \dots, v_n are even and $\text{sign}(v_1, \dots, v_n; \sigma) = \text{sign}(\sigma)$ is the usual signum of the permutation for all v_1, \dots, v_n odd. It is then straightforward to check that

$$(v_1 \otimes \dots \otimes v_n) \triangleleft \sigma = \text{sign}(v_1, \dots, v_n; \sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)} \quad (2.2)$$

extends to a well-defined right action of S_n on $V^{\otimes n}$. We use this right action to define the symmetrization operator

$$\mathcal{S}_n: V^{\otimes n} \ni v \mapsto \mathcal{S}_n(v) = \frac{1}{n!} \sum_{\sigma \in S_n} v \triangleleft \sigma \in V^{\otimes n}. \quad (2.3)$$

One has $\mathcal{S}_n \circ \mathcal{S}_n = \mathcal{S}_n$ since (2.2) is an action. In the case where $V = V_0$, the operator \mathcal{S}_n is the usual total symmetrization, if $V = V_1$ we get the total antisymmetrization operator.

For later use it will be advantageous to define the symmetric algebra not as a quotient algebra of the tensor algebra but as a subspace with a new product. Thus we set

$$S^n(V) = \text{im } \mathcal{S}_n \subseteq V^{\otimes n} \quad \text{and} \quad S^0(V) = \mathbb{k}. \quad (2.4)$$

The elements in $S^n(V)$ consist of the symmetric, i.e. invariant tensors with respect to the action (2.2). Moreover, we set

$$S^\bullet(V) = \bigoplus_{n=0}^{\infty} S^n(V) \subseteq T^\bullet(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}. \quad (2.5)$$

Alternatively, we can use the idempotent $\mathcal{S} = \bigoplus_{n=0}^{\infty} \mathcal{S}_n$ where $\mathcal{S}_0 = \text{id}$ and get $S^\bullet(V) = \text{im } \mathcal{S}$.

Next we define the symmetric tensor product $\mu: S^\bullet(V) \otimes S^\bullet(V) \longrightarrow S^\bullet(V)$ of $v, w \in S^\bullet(V)$ as usual by

$$vw = \mu(v \otimes w) = \mathcal{S}(v \otimes w). \quad (2.6)$$

Most of the time we shall omit the symbol μ for products. Then the following statement is well-known:

Lemma 2.1 *The symmetric tensor product turns $S^\bullet(V)$ into an associative commutative unital algebra freely generated by V .*

Moreover, it is \mathbb{Z} -graded with respect to the tensor degree. Note however, that we do not use this degree for sign purposes at all. Freely generated means that a homogeneous map $\phi: V \rightarrow \mathcal{A}$ of parity $\mathbf{0}$ into another associative \mathbb{Z}_2 -graded commutative unital algebra \mathcal{A} has a unique extension $\Phi: \mathbf{S}^\bullet(V) \rightarrow \mathcal{A}$ as unital algebra homomorphism.

Beside the symmetric algebra we will also need tensor products of algebras. Thus let \mathcal{A} and \mathcal{B} be two associative \mathbb{Z}_2 -graded algebras. On their tensor product $\mathcal{A} \otimes \mathcal{B}$ a new product is defined by linear extension of

$$(a \otimes b)(a' \otimes b') = (-1)^{ba'} aa' \otimes bb', \quad (2.7)$$

where $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ are homogeneous elements. This turns $\mathcal{A} \otimes \mathcal{B}$ again into an associative \mathbb{Z}_2 -graded algebra.

Finally, we recall that the Koszul sign rule also applies to tensor products of maps and evaluations, i.e. for homogeneous maps $\phi: V \rightarrow W$ and $\psi: \tilde{V} \rightarrow \tilde{W}$ we define their tensor product $\phi \otimes \psi: V \otimes \tilde{V} \rightarrow W \otimes \tilde{W}$ by

$$(\phi \otimes \psi)(v \otimes w) = (-1)^{\psi v} \phi(v) \otimes \psi(w) \quad (2.8)$$

on homogeneous vectors and extend linearly.

2.2 Constant Poisson Structures and Formal Star Products

Recall that a homogeneous multilinear map of parity $\mathbf{0}$

$$P: \mathbf{S}^\bullet(V) \times \cdots \times \mathbf{S}^\bullet(V) \rightarrow \mathbf{S}^\bullet(V) \quad (2.9)$$

is called a *multiderivation* if for each argument it satisfies the Leibniz rule

$$\begin{aligned} & P(v_1, \dots, v_{k-1}, v_k v'_k, v_{k+1}, \dots, v_n) \\ &= (-1)^{(v_1 + \cdots + v_{k-1})v_k} v_k P(v_1, \dots, v'_k, \dots, v_n) + (-1)^{v'_k(v_{k+1} + \cdots + v_n)} P(v_1, \dots, v_k, \dots, v_n) v'_k, \end{aligned} \quad (2.10)$$

where we follow again the Koszul sign rule. Note that if one v_i is a constant, i.e. $v_i \in \mathbf{S}^0(V)$, then $P(v_1, \dots, v_n) = 0$. Since V generates $\mathbf{S}^\bullet(V)$, a multiderivation is uniquely determined by its values on $V \times \cdots \times V$. Conversely, any multilinear homogeneous map $V \times \cdots \times V \rightarrow \mathbf{S}^\bullet(V)$ of parity $\mathbf{0}$ extends to a multiderivation since V generates $\mathbf{S}^\bullet(V)$ *freely*. Though one can also consider odd multiderivations, we shall not need them in the sequel.

For later estimates, we will need the following more explicit form of this extension for the particular case of a *bilinear* homogeneous map

$$\Lambda: V \times V \rightarrow \mathbb{k} = \mathbf{S}^0(V) \quad (2.11)$$

of parity $\mathbf{0}$. To this end we first define the linear map

$$P_\Lambda: \mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V) \rightarrow \mathbf{S}^{\bullet-1}(V) \otimes \mathbf{S}^{\bullet-1}(V) \quad (2.12)$$

to be the linear extension of

$$\begin{aligned} & P_\Lambda(v_1 \dots v_n \otimes w_1 \dots w_m) \\ &= \sum_{k=1}^n \sum_{\ell=1}^m (-1)^{v_k(v_{k+1} + \cdots + v_n) + w_\ell(w_1 + \cdots + w_{\ell-1})} \Lambda(v_k, w_\ell) v_1 \cdots \overset{k}{\wedge} \cdots v_n \otimes w_1 \cdots \overset{\ell}{\wedge} \cdots w_m. \end{aligned} \quad (2.13)$$

The requirement on the \mathbb{Z} -grading implies that P_Λ vanishes on tensors of the form $\mathbb{1} \otimes w$ or $v \otimes \mathbb{1}$.

First note that Λ is of parity $\mathbf{0}$ and thus $\Lambda(v_k, w_\ell)$ is only nontrivial if v_k and w_ℓ have the same parity. Hence we can exchange the parities v_k and w_ℓ in the above sign. Second, note that the map P_Λ is indeed well-defined since the right hand side is totally symmetric in v_1, \dots, v_n and in w_1, \dots, w_m . With respect to the algebra structure (2.7) of $\mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V)$ we can characterize the map P_Λ now as follows:

Lemma 2.2 *The map P_Λ is the unique map with*

- i.) $P_\Lambda(v \otimes w) = \Lambda(v, w)\mathbb{1} \otimes \mathbb{1}$ for all $v, w \in V$,*
- ii.) $P_\Lambda(v \otimes wu) = P_\Lambda(v \otimes w)(\mathbb{1} \otimes u) + (-1)^{wv}(\mathbb{1} \otimes w)P_\Lambda(v \otimes u)$ for all $v, w, u \in \mathbf{S}^\bullet(V)$,*
- iii.) $P_\Lambda(vw \otimes u) = (v \otimes \mathbb{1})P_\Lambda(w \otimes u) + (-1)^{wu}P_\Lambda(v \otimes u)(\mathbb{1} \otimes w)$ for all $v, w, u \in \mathbf{S}^\bullet(V)$.*

Note that the two Leibniz rules imply $P_\Lambda(v \otimes \mathbb{1}) = 0 = P_\Lambda(\mathbb{1} \otimes v)$ for all $v \in \mathbf{S}^\bullet(V)$.

In order to rewrite the Leibniz rules for P_Λ in a more conceptual way, we have to introduce the canonical flip operator $\tau_{VW}: V \otimes W \longrightarrow W \otimes V$ for \mathbb{Z}_2 -graded vector spaces V and W by

$$\tau_{VW}(v \otimes w) = (-1)^{vw} w \otimes v \quad (2.14)$$

on homogeneous elements and linear extension to all tensors. We usually write τ if the reference to the underlying vector spaces is clear. Using τ we define the operators

$$P_\Lambda^{12}, P_\Lambda^{23}, P_\Lambda^{13}: \mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V) \longrightarrow \mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V) \quad (2.15)$$

on the triple tensor product by

$$P_\Lambda^{12} = P_\Lambda \otimes \text{id}, \quad P_\Lambda^{23} = \text{id} \otimes P_\Lambda, \quad \text{and} \quad P_\Lambda^{13} = (\text{id} \otimes \tau) \circ (P_\Lambda \otimes \text{id}) \circ (\text{id} \otimes \tau). \quad (2.16)$$

These operators have again parity $\mathbf{0}$ and change the tensor degrees by $(-1, -1, 0)$, $(0, -1, -1)$, and by $(-1, 0, -1)$, respectively.

Lemma 2.3 *The Leibniz rules for P_Λ can be written as*

$$P_\Lambda \circ (\mu \otimes \text{id}) = (\mu \otimes \text{id}) \circ (P_\Lambda^{13} + P_\Lambda^{23}) \quad (2.17)$$

and

$$P_\Lambda \circ (\text{id} \otimes \mu) = (\text{id} \otimes \mu) \circ (P_\Lambda^{12} + P_\Lambda^{13}). \quad (2.18)$$

Analogously, we have similar Leibniz rules for the operators P_Λ^{12} , P_Λ^{23} , and P_Λ^{13} which show that they will be uniquely determined by their values on generators of $\mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V)$. Hence the products $P_\Lambda^{12} \circ P_\Lambda^{23}$ etc. will be uniquely determined by their values on quadratic expressions in the generators. This will allow for a rather straightforward computation leading to the following observation:

Lemma 2.4 *The operators P_Λ^{12} , P_Λ^{23} , and P_Λ^{13} commute pairwise.*

This lemma together with the Leibniz rule in form of Lemma 2.3 gives immediately the following result, see e.g. [24, Sect. 6.2.4] for a detailed proof:

Proposition 2.5 *On $\mathbf{S}^\bullet(V)[[\nu]]$ one obtains a \mathbb{Z}_2 -graded associative $\mathbb{k}[[\nu]]$ -bilinear multiplication by*

$$v \star_{\nu\Lambda} w = \mu \circ e^{\nu P_\Lambda}(v \otimes w), \quad (2.19)$$

where all \mathbb{k} -multilinear maps are extended to be $\mathbb{k}[[\nu]]$ -multilinear as usual.

Remark 2.6 (Commuting derivations) A particularly simple case is obtained for $\Lambda = \varphi \otimes \psi$ with $\varphi, \psi \in V^*$ of equal parity. In this case, we denote by $X_\varphi, X_\psi: \mathbf{S}^\bullet(V) \longrightarrow \mathbf{S}^{\bullet-1}(V)$ the corresponding derivations of the same parity. It is then easy to see that $[X_\varphi, X_\psi] = 0$ and that $P_\Lambda = X_\varphi \otimes X_\psi$. This example (without \mathbb{Z}_2 -grading) of a formal associative deformation via commuting derivations was first considered by Gerstenhaber in [17, Thm. 8] and was generalized in various ways ever since.

Also the next proposition is folklore and easily verified:

Proposition 2.7 *Let \mathcal{A} be an associative \mathbb{Z}_2 -graded commutative algebra and let \star be a formal associative deformation of it such that $(\mathcal{A}[[\nu]], \star)$ is still \mathbb{Z}_2 -graded. Then the first order of the \star -commutator defines a Poisson bracket on \mathcal{A} .*

Of course, we always take the \mathbb{Z}_2 -graded commutators and Poisson brackets. In our example, this leads to the following Poisson bracket:

Corollary 2.8 *Let Λ be as above and set $P_\Lambda^{\text{opp}} = \tau \circ P_\Lambda \circ \tau$. Then*

$$\{v, w\}_\Lambda = \mu \circ (P_\Lambda - P_\Lambda^{\text{opp}})(a \otimes b) \quad (2.20)$$

defines a Poisson bracket for $S^\bullet(V)$.

Alternatively, we can also consider the *symmetric* and *antisymmetric part*

$$\Lambda_\pm = \frac{1}{2}(\Lambda \pm \Lambda \circ \tau) \quad (2.21)$$

of Λ such that $\Lambda = \Lambda_+ + \Lambda_-$. Then we note that with $\Lambda^{\text{opp}} = \Lambda \circ \tau$ we have

$$P_{\Lambda^{\text{opp}}} = P_\Lambda^{\text{opp}} \quad (2.22)$$

and thus

$$P_\Lambda - P_\Lambda^{\text{opp}} = 2P_{\Lambda_-}. \quad (2.23)$$

Thus $\{v, w\}_\Lambda = 2\mu \circ P_{\Lambda_-}(v \otimes w)$ depends only on the antisymmetric part. Nevertheless, the star product \star in (2.19) depends on Λ and not just on Λ_- . It is this Poisson bracket for which $\star_{\nu\Lambda}$ provides a formal deformation quantization.

In general, one requires only a formal star product but since our Poisson bracket is rather particular, we can sharpen the deformation result as follows:

Corollary 2.9 *The product $\star_{\nu\Lambda}$ restricts to $S^\bullet(V)[\nu]$ which becomes an associative \mathbb{Z}_2 -graded algebra over $\mathbb{k}[\nu]$.*

More precisely, for $v, w \in S^\bullet(V)$ we have $P_\Lambda^n(v \otimes w) = 0$ as soon as $n \in \mathbb{N}_0$ is larger than the maximal symmetric degree in v or w . It follows that in $S^\bullet(V)[\nu]$ we can replace the formal parameter ν by any element of \mathbb{k} and get a well-defined associative multiplication from $\star_{\nu\Lambda}$.

Also the following result is well-known and obtained from an easy induction: the elements of V generate $S^\bullet(V)[\nu]$ with respect to $\star_{\nu\Lambda}$:

Corollary 2.10 *The $\mathbb{k}[\nu]$ -algebra $S^\bullet(V)[\nu]$ is generated by V .*

2.3 Symmetries and Equivalences

The symmetric algebra $S^\bullet(V)$ can be interpreted as the polynomials on the “predual” of V , which, of course, needs not to exist in infinite dimensions. Alternatively, $S^\bullet(V)$ injects as a subalgebra into the polynomials $\text{Pol}^\bullet(V^*)$ on the dual of V . We use this heuristic point of view now to establish some symmetries of $\{\cdot, \cdot\}_\Lambda$ and $\star_{\nu\Lambda}$ which justify the term “constant” Poisson structure.

Let $\varphi \in V^*$ be an even linear functional, i.e. $\varphi \in V_0^*$, then the linear map $v \mapsto v + \varphi(v)\mathbb{1}$ is even, too, and thus it extends uniquely to a unital algebra homomorphism $\tau_\varphi^*: T^\bullet(V) \rightarrow T^\bullet(V)$. Clearly, the symmetry properties of the tensors in $T^\bullet(V)$ are preserved by τ_φ^* and thus it restricts to a unital algebra homomorphism

$$\tau_\varphi^*: S^\bullet(V) \rightarrow S^\bullet(V), \quad (2.24)$$

now with respect to the symmetric tensor product. Clearly, $\tau_0^* = \text{id}$ and $\tau_\varphi^* \tau_\psi^* = \tau_{\varphi+\psi}^*$ for all $\varphi, \psi \in V_0^*$. Thus we get an action of the abelian group V_0^* on $S^\bullet(V)$ by automorphisms. In the interpretation of

polynomials these automorphisms correspond to pull-backs with *translations* via φ , hence the above notation.

The other important symmetry emerges from the endomorphisms of V itself. Let $A: V \rightarrow V$ be an even linear map and denote the extension as unital algebra homomorphism again by $A: S^\bullet(V) \rightarrow S^\bullet(V)$. This yields an embedding of $\text{End}_0(V)$ into the unital algebra endomorphisms of $S^\bullet(V)$. In particular, we get a group homomorphism of $\text{GL}_0(V)$ into $\text{Aut}_0(S^\bullet(V))$. For $A \in \text{GL}_0(V)$ and $\varphi \in V_0^*$ we have the relation $A^{-1}\tau_\varphi^*Av = \tau_{A^*\varphi}^*v$ for the generators $v \in V$ and hence also in general

$$A^{-1}\tau_\varphi^*A = \tau_{A^*\varphi}^*. \quad (2.25)$$

This gives an action of the semidirect product $\text{GL}_0(V) \ltimes V^*$ on $S^\bullet(V)$ via unital algebra automorphisms.

For the bilinear map Λ we consider the group of invertible even endomorphisms of V preserving it and denote this group by

$$\text{Aut}(V, \Lambda) = \{A \in \text{GL}_0(V) \mid \Lambda(Av, Aw) = \Lambda(v, w) \text{ for all } v, w \in V\}. \quad (2.26)$$

Note that such an automorphism preserves Λ_+ and Λ_- separately. However, Λ_- and Λ_+ might have a larger invariance group than $\text{Aut}(V, \Lambda)$.

Remark 2.11 Suppose that $\Lambda = -\Lambda^{\text{opp}} = \Lambda_-$ is already antisymmetric and non-degenerate. In the case where $V = V_0$ consists of even vectors only, Λ is a symplectic form and $\text{Aut}(V, \Lambda)$ is the corresponding symplectic group. In the case where $V = V_1$ is odd, Λ corresponds to an inner product and $\text{Aut}(V, \Lambda)$ is the corresponding pseudo-orthogonal group. Note however, that we will also be interested in the case where Λ is not necessarily antisymmetric and not necessarily non-degenerate.

Lemma 2.12 *The subgroup $\text{Aut}(V, \Lambda) \ltimes V^*$ acts on $S^\bullet(V)$ as automorphisms of $\{\cdot, \cdot\}_\Lambda$ and \star .*

Proof. First consider $A \in \text{Aut}(V, \Lambda)$. Then on generators one sees that $P_\Lambda \circ (A \otimes A) = (A \otimes A) \circ P_\Lambda$, which therefore holds in general. From this we see that A is an automorphism of both, the Poisson bracket and the star product. Analogously, for $\varphi \in V_0^*$ one checks first on generators and then in general that $P_\Lambda \circ (\tau_\varphi^* \otimes \tau_\varphi^*) = (\tau_\varphi^* \otimes \tau_\varphi^*) \circ P_\Lambda$. \square

In this sense, both the Poisson bracket and the star product are *constant*, i.e. translation-invariant.

In a next step we discuss to what extent the automorphisms are inner. We consider only the infinitesimal picture as the integrated version will require analytical tools. The bilinear form Λ induces a linear map into the dual V^* . More precisely, we need the antisymmetric part Λ_- of Λ as it appears also in the Poisson bracket (2.20). This defines an even linear map

$$\sharp: V \ni v \mapsto v^\sharp = \Lambda_-(v, \cdot) \in V^*. \quad (2.27)$$

Lemma 2.13 *Let $\varphi \in V^*$ be homogeneous and denote by $X_\varphi: S^\bullet(V) \rightarrow S^\bullet(V)$ the homogeneous derivation extending $\varphi: V \rightarrow \mathbb{k}$.*

i.) X_φ is a Poisson derivation of parity φ , i.e. we have

$$X_\varphi\{a, b\}_\Lambda = \{X_\varphi(a), b\}_\Lambda + (-1)^{\varphi a}\{a, X_\varphi(b)\}_\Lambda \quad (2.28)$$

for all homogeneous $a, b \in S^\bullet(V)$.

ii.) X_φ is inner iff $\varphi \in \text{im } \sharp$. In this case $X_\varphi = \{v, \cdot\}_\Lambda$ for any $v \in V$ with $2v^\sharp = \varphi$.

iii.) X_φ is a derivation of $\star_{\nu\Lambda}$, i.e. we have

$$X_\varphi(a \star_{\nu\Lambda} b) = X_\varphi(a) \star_{\nu\Lambda} b + (-1)^{\varphi a} a \star_{\nu\Lambda} X_\varphi(b) \quad (2.29)$$

for all homogeneous $a, b \in \mathbf{S}^\bullet(V)$.

iv.) X_φ is a quasi-inner derivation of $\star_{\nu\Lambda}$, i.e. $X_\varphi = \frac{1}{\nu}[a, \cdot]_{\star_{\nu\Lambda}}$ for some $a \in \mathbf{S}^\bullet(V)[[\nu]]$, iff $\varphi \in \text{im } \sharp$. In this case $a = v \in V$ with $2v^\sharp = \varphi$ will do the job.

Proof. Consider an even linear map $P: \mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V) \longrightarrow \mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V)$ satisfying the Leibniz rules from Lemma 2.2, ii.) and iii.), and let X be any homogeneous derivation of either even or odd parity. Then we claim that the operator

$$D = P \circ (X \otimes \text{id} + \text{id} \otimes X) - (X \otimes \text{id} + \text{id} \otimes X) \circ P$$

satisfies the Leibniz rules

$$D(ab \otimes c) = (-1)^{bc} D(a \otimes c)(b \otimes \mathbb{1}) + (-1)^{Xa}(a \otimes \mathbb{1})D(b \otimes c)$$

and

$$D(a \otimes bc) = D(a \otimes b)(\mathbb{1} \otimes c) + (-1)^{(X+a)b}(\mathbb{1} \otimes b)D(a \otimes c)$$

for all homogeneous $a, b, c \in \mathbf{S}^\bullet(V)$. This is a simple verification and does not use that P is (anti-)symmetric. In our case, we conclude that D is uniquely determined by its values on the generators of $\mathbf{S}^\bullet(V) \otimes \mathbf{S}^\bullet(V)$. For $P = P_\Lambda$ and $X = X_\varphi$ it is easy to check that $D = 0$ on generators and thus P_Λ and $(X_\varphi \otimes \text{id} + \text{id} \otimes X_\varphi)$ commute. But this implies the first as well as the third part. Now consider $\varphi \in \text{im } \sharp$, i.e. there is a $v \in V$ with $\varphi = 2\Lambda_-(v, \cdot)$. In this case we get for $w \in V$

$$\{v, w\}_\Lambda = \Lambda(v, w)\mathbb{1} - (-1)^{vw}\Lambda(w, v)\mathbb{1} = 2\Lambda_-(v, w)\mathbb{1} = \varphi(w)\mathbb{1} = X_\varphi(w).$$

Since the derivation X_φ is determined by its values on generators this implies $X_\varphi = \{v, \cdot\}_\Lambda$. For the converse, assume that $X_\varphi = \{v, \cdot\}_\Lambda$ for some $v \in \mathbf{S}^\bullet(V)$ which we write as $v = \sum_n v_n$ with $v_n \in \mathbf{S}^n(V)$. Then for $w \in V$ we have $\{v_n, w\}_\Lambda \in \mathbf{S}^{n-1}(V)$ while $X_\varphi(w) \in \mathbf{S}^0(V)$. Thus we necessarily have $X_\varphi(w) = \{v_1, w\}_\Lambda$, i.e. the higher order terms in v are not necessary. But then $X_\varphi = \{v_1, \cdot\}_\Lambda$ follows, proving $\varphi = 2v_1^\sharp$. The fourth part is similar, since for $v \in V$ we have $v \star_{\nu\Lambda} a = va + \nu \mu \circ P_\Lambda(v \otimes a)$ without higher order terms. Thus $[v, a]_\star = \nu\{v, a\}_\Lambda$ and we can argue as in the second part. \square

Remark 2.14 We see here a notorious difficulty of formal star products: the derivation generating a symmetry is only quasi-inner and not inner. Thus a naive exponentiation of the generating element would lead us outside the formal power series. In fact, algebraically it can not be well-defined at all. Thus the symmetry τ_φ^* is an *outer* automorphism. We shall come back to this when we have some more analytical framework.

We can extend the results of Lemma 2.12 in the following way: suppose we have two vector spaces V and W with two bilinear forms Λ_V and Λ_W on them. Then an even linear map $A: V \longrightarrow W$ is called a *Poisson map* if

$$\Lambda_W(A(v), A(v')) = \Lambda_V(v, v') \quad (2.30)$$

for all $v, v' \in V$. The induced map $A: \mathbf{S}^\bullet(V) \longrightarrow \mathbf{S}^\bullet(W)$ is then easily shown to satisfy $P_{\Lambda_W} \circ (A \otimes A) = (A \otimes A) \circ P_{\Lambda_V}$, generalizing the computation in the proof of Lemma 2.12 slightly. From this we see that A is a homomorphism of Poisson algebras and star product algebras. Thus we arrive at the following simple functoriality statement:

Proposition 2.15 *The construction of $\{\cdot, \cdot\}_\Lambda$ and $\star_{\nu\Lambda}$ is functorial with respect to Poisson maps.*

Let us now discuss how we can change the star product by changing the *symmetric* part Λ_+ of Λ as in (2.21). Symmetry means that $\Lambda_+(v, w) = (-1)^{vw} \Lambda_+(w, v)$ for homogeneous elements in V .

Let $g: V \times V \rightarrow \mathbb{k}$ be another symmetric and even bilinear form, which we can think of as a \mathbb{Z}_2 -graded version of an inner product. We define now a second order “Laplacian” associated to g as follows. For homogeneous vectors $v_1, \dots, v_n \in V$ we set

$$\Delta_g(v_1 \cdots v_n) = \sum_{i < j} (-1)^{v_i(v_1 + \cdots + v_{i-1})} (-1)^{v_j(v_1 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_{j-1})} g(v_i, v_j) v_1 \cdots \overset{i}{\wedge} \cdots \overset{j}{\wedge} \cdots v_n, \quad (2.31)$$

and extend this again by linearity to an operator

$$\Delta_g: S^\bullet(V) \rightarrow S^{\bullet-2}(V). \quad (2.32)$$

Note that Δ_g has even parity since g vanishes on vectors of different parities. This is no longer a derivation but a second order differential operator. More precisely, we have the following “Leibniz rule” for Δ_g :

Lemma 2.16 *The operator Δ_g satisfies*

$$\Delta_g \circ \mu = \mu \circ (\Delta_g \otimes \text{id} + P_g + \text{id} \otimes \Delta_g). \quad (2.33)$$

Proof. On $v_1 \cdots v_n \otimes w_1 \cdots w_m$ with homogeneous vectors $v_1, \dots, v_n, w_1, \dots, w_m \in V$ this is just a straightforward computation. \square

Lemma 2.17 *Let $\Lambda, \Lambda', g: V \times V \rightarrow \mathbb{k}$ be even bilinear maps and let g be symmetric. Then the operators $\Delta_g \otimes \text{id}$, $\text{id} \otimes \Delta_g$, P_Λ , and $P_{\Lambda'}$ commute pairwise.*

Proof. Again, one just checks this on $v_1 \cdots v_n \otimes w_1 \cdots w_m$ for homogeneous vectors $v_1, \dots, v_n, w_1, \dots, w_m \in V$ which is a lengthy but straightforward computation, the details of which we shall omit. \square

We use these commutation relations now to prove the following equivalence statement: the isomorphism class of the deformation depends only on the *antisymmetric* part of Λ .

Proposition 2.18 *Let $\Lambda, \Lambda': V \times V \rightarrow \mathbb{k}$ be two even bilinear forms on V such that their antisymmetric parts $\Lambda_- = \Lambda'_-$ coincide. Then the corresponding star products $\star_{\nu\Lambda}$ and $\star_{\nu\Lambda'}$ are equivalent via the equivalence transformation*

$$e^{\nu\Delta_g}(a \star_{\nu\Lambda} b) = (e^{\nu\Delta_g} a) \star_{\nu\Lambda'} (e^{\nu\Delta_g} b) \quad (2.34)$$

for all $a, b \in S^\bullet(V)[[\nu]]$ where $g = \Lambda' - \Lambda = \Lambda'_+ - \Lambda_+$.

Proof. The proof is now fairly easy. Analogously to [24, Exercise 5.7] we have

$$\begin{aligned} e^{\nu\Delta_g}(a \star_{\nu\Lambda} b) &= e^{\nu\Delta_g} \circ \mu \circ e^{\nu P_\Lambda}(a \otimes b) \\ &= \mu \circ e^{\nu(\Delta_g \otimes \text{id} + P_g + \text{id} \otimes \Delta_g)} \circ e^{\nu P_\Lambda}(a \otimes b) \\ &= \mu \circ e^{\nu(P_\Lambda + P_g)} \circ (e^{\nu\Delta_g} \otimes e^{\nu\Delta_g})(a \otimes b) \\ &= \mu \circ e^{\nu P_{\Lambda'}} \circ (e^{\nu\Delta_g} a \otimes e^{\nu\Delta_g} b), \end{aligned}$$

since $P_\Lambda + P_g = P_{\Lambda+g}$ and since $\Lambda + g = \Lambda'$. Note that g is indeed symmetric. \square

3 Continuity of the Star Product

In this section we establish a locally convex topology on $S^\bullet(V)$ for which the formal star product, after substituting the formal parameter by a real or complex number z , will be continuous. Starting point is a locally convex topology on V , which we will assume to be Hausdorff, and a continuity assumption on Λ . From now on the field of scalars \mathbb{K} is either \mathbb{R} or \mathbb{C} .

3.1 The Topology for $S^\bullet(V)$

Let V be now a real or complex \mathbb{Z}_2 -graded Hausdorff locally convex vector space. We require that the grading is *compatible* with the topological structure, i.e. the projections onto the even and odd parts in $V = V_0 \oplus V_1$ are continuous. Thus we have for every continuous seminorm p on V another continuous seminorm q with $p(v_0), p(v_1) \leq q(v)$ for all $v \in V$. This implies that the even and odd part of V constitute complementary closed subspaces.

In principle, there are many interesting locally convex topologies on $S^\bullet(V)$ induced by the one on V . We shall construct now a rather particular one.

First we will endow the tensor products $V^{\otimes n}$ with the π -topology. Recall that for seminorms p_1, \dots, p_n on V one defines the seminorm $p_1 \otimes \dots \otimes p_n$ on $V^{\otimes n}$ by

$$(p_1 \otimes \dots \otimes p_n)(v) = \inf \left\{ \sum_i p_1(v_i^{(1)}) \dots p_n(v_i^{(n)}) \mid v = \sum_i v_i^{(1)} \otimes \dots \otimes v_i^{(n)} \right\}, \quad (3.1)$$

where the infimum is taken over all possibilities to write the tensor v as a linear combination of elementary (i.e. factorizing) tensors. One has $(p_1 \otimes \dots \otimes p_n) \otimes (q_1 \otimes \dots \otimes q_m) = p_1 \otimes \dots \otimes p_n \otimes q_1 \otimes \dots \otimes q_m$ and on factorizing tensors one gets $(p_1 \otimes \dots \otimes p_n)(v_1 \otimes \dots \otimes v_n) = p_1(v_1) \dots p_n(v_n)$. We shall use the abbreviation $p^n = p \otimes \dots \otimes p$ for n copies of the same seminorm p , where p^0 is the usual absolute value on \mathbb{K} .

The π -topology on $V^{\otimes n}$ is obtained by taking all seminorms of the form $p_1 \otimes \dots \otimes p_n$ with p_1, \dots, p_n being continuous seminorms on V . Equivalently, one can take all p^n with p being a continuous seminorm on V . Analogously, one defines the π -topology for the tensor products of different locally convex spaces. We denote the tensor product endowed with the π -topology by \otimes_π . It is clear that the induced \mathbb{Z}_2 -grading of $V^{\otimes n}$ is again compatible with the π -topology. More explicitly, we have the following statement:

Lemma 3.1 *Let p be a continuous seminorm on V and choose a continuous seminorm q such that $p(v_0), p(v_1) \leq q(v)$. Then for all $v \in V^{\otimes n}$ one has*

$$p^n(v_0), p^n(v_1) \leq q^n(v). \quad (3.2)$$

For concrete estimates the following simple lemma is useful: it suffices to check estimates on factorizing tensors only:

Lemma 3.2 *Let V_1, \dots, V_n, W be vector spaces and let $\phi: V_1 \times \dots \times V_n \rightarrow W$ be an n -linear map, identified with a linear map $\phi: V_1 \otimes \dots \otimes V_n \rightarrow W$ as usual. If p_1, \dots, p_n, q are seminorms on V_1, \dots, V_n, W , respectively, such that for all $v_1 \in V_1, \dots, v_n \in V_n$ one has*

$$q(\phi(v_1, \dots, v_n)) \leq p_1(v_1) \dots p_n(v_n), \quad (3.3)$$

then one has for all $v \in V_1 \otimes \dots \otimes V_n$

$$q(\phi(v)) \leq (p_1 \otimes \dots \otimes p_n)(v). \quad (3.4)$$

Since we view the symmetric powers $S^n(V)$ as subspace of $V^{\otimes n}$ we can inherit the π -topology also for $S^n(V)$, indicated by $S_\pi^n(V)$. Then we get the following simple properties of the symmetric tensor product:

Lemma 3.3 *Let $n, m \in \mathbb{N}_0$ and let p be a continuous seminorm on V .*

i.) The symmetrizer $S_n: V^{\otimes \pi n} \rightarrow V^{\otimes \pi n}$ is continuous and for all $v \in V^{\otimes \pi n}$ one has

$$p^n(S_n(v)) \leq p^n(v). \quad (3.5)$$

ii.) $S_\pi^n(V) \subseteq V^{\otimes \pi n}$ is a closed subspace.

iii.) For $v \in S^n(V)$ and $w \in S^m(V)$ one has

$$p^{n+m}(vw) \leq p^n(v) p^m(w). \quad (3.6)$$

Proof. The first part is clear for factorizing tensors and hence Lemma 3.2 applies. The second follows as $S^n(V) = \ker(\text{id} - S_n)$ by definition. The third is clear from the definition and from (3.5). \square

On the tensor algebra $T^\bullet(V)$ there are at least two canonical locally convex topologies: the Cartesian product topology inherited from $\prod_{n=0}^\infty V^{\otimes \pi n}$ and the direct sum topology which is the inductive limit topology of the finite direct sums. While the first is very coarse, the second is very fine. Nevertheless, both of them induce the π -topology on each subspace $V^{\otimes n}$. We are now searching for something in between.

We fix a parameter $R \in \mathbb{R}$ and consider for a given continuous seminorm p on V the new seminorm

$$p_{R,1}(v) = \sum_{n=0}^\infty p^n(v_n) n!^R \quad (3.7)$$

on the tensor algebra $T^\bullet(V)$, where we write $v = \sum_{n=0}^\infty v_n$ as the sum of its components with fixed tensor degree $v_n \in V^{\otimes n}$. Analogously, we define

$$p_{R,\infty}(v) = \sup_{n \in \mathbb{N}_0} \{p^n(v_n) n!^R\}. \quad (3.8)$$

In principle, we have also ℓ^p -versions for all $p \in [1, \infty)$, but the above two extreme cases will suffice for the following.

The seminorms control the growth of the contributions $p^n(v_n)$ for $n \rightarrow \infty$ compared to a power of $n!$ which we can view as weights from a weighted counting measure. The choice of the factorials as weights will become clear later. We list some first elementary properties of these seminorms.

Lemma 3.4 *Let p and q be seminorms on V and $R, R' \in \mathbb{R}$.*

i.) One has for all $v \in T^\bullet(V)$

$$p_{R,\infty}(v) \leq p_{R,1}(v). \quad (3.9)$$

ii.) If $R' > R$ then there is a constant $c > 0$ such that for all $v \in T^\bullet(V)$ one has

$$p_{R,1}(v) \leq c p_{R',\infty}(v). \quad (3.10)$$

iii.) Both seminorms $p_{R,1}$ and $p_{R,\infty}$ restrict to $n!^R p^n$ on $V^{\otimes n}$.

iv.) If $q \leq p$ then $q_{R,1} \leq p_{R,1}$ and $q_{R,\infty} \leq p_{R,\infty}$.

v.) If $R' > R$ then $p_{R,1}(v) \leq p_{R',1}(v)$ and $p_{R,\infty}(v) \leq p_{R',\infty}(v)$ for all $v \in T^\bullet(V)$.

Proof. The parts *i.)*, *iii.)*, *iv.)*, and *v.)* are clear. For the second we have

$$p_{R,1}(v) = \sum_{n=0}^\infty p^n(v_n) n!^R \leq \sup_{n \in \mathbb{N}_0} p^n(v_n) n!^{R'} \underbrace{\sum_{n=0}^\infty \frac{1}{n!^{R'-R}}}_c,$$

with a convergent series $c < \infty$ as soon as $R' - R > 0$. \square

The seemingly trivial second part will have an important consequence later when we discuss the nuclearity properties of the Weyl algebra.

We use now all the seminorms $p_{R,1}$ for a fixed R to define a new locally convex topology on the tensor algebra:

Definition 3.5 *Let $R \in \mathbb{R}$. Then $T_{R,1}^\bullet(V)$ is the tensor algebra of V equipped with the locally convex topology determined by all the seminorms $p_{R,1}$ with p running through all continuous seminorms on V . Analogously, one defines $T_{R,\infty}^\bullet(V)$ using all the seminorms $p_{R,\infty}$ instead.*

In the following, we will mainly be interested in the case of positive R where we have a *decay* of the numbers $p^n(v_n)$.

Lemma 3.6 *Let $R' > R \geq 0$.*

i.) The tensor product is continuous on $T_{R,1}^\bullet(V)$. More precisely, one has

$$p_{R,1}(v \otimes w) \leq (2^R p)_{R,1}(v)(2^R p)_{R,1}(w) \quad (3.11)$$

for all $v, w \in T_{R,1}^\bullet(V)$.

ii.) For all $n \in \mathbb{N}_0$ the projections and the inclusions

$$T_{R,1}^\bullet(V) \longrightarrow V^{\otimes \pi^n} \longrightarrow T_{R,1}^\bullet(V) \quad (3.12)$$

are continuous.

iii.) The completions $\hat{T}_{R,1}^\bullet(V)$ and $\hat{T}_{R,\infty}^\bullet(V)$ of $T_{R,1}^\bullet(V)$ and $T_{R,\infty}^\bullet(V)$ can explicitly be described by

$$\hat{T}_{R,1}^\bullet(V) = \left\{ v = \sum_{n=0}^{\infty} v_n \mid p_{R,1}(v) < \infty \text{ for all } p \right\} \subseteq \prod_{n=0}^{\infty} V^{\hat{\otimes} \pi^n} \quad (3.13)$$

and

$$\hat{T}_{R,\infty}^\bullet(V) = \left\{ v = \sum_{n=0}^{\infty} v_n \mid p_{R,\infty}(v) < \infty \text{ for all } p \right\} \subseteq \prod_{n=0}^{\infty} V^{\hat{\otimes} \pi^n}, \quad (3.14)$$

where p runs through all continuous seminorms of V and we extend $p_{R,1}$ and $p_{R,\infty}$ to the Cartesian product by allowing the value $+\infty$.

iv.) We have the continuous inclusions

$$\hat{T}_{R',\infty}^\bullet(V) \longrightarrow \hat{T}_{R,1}^\bullet(V) \longrightarrow \hat{T}_{R,\infty}^\bullet(V). \quad (3.15)$$

v.) The \mathbb{Z}_2 -grading of $T_{R,1}^\bullet(V)$ is continuous. More explicitly, if p and q are continuous seminorms with $p(v_0), p(v_1) \leq q(v)$ for all $v \in V$ then we have

$$p_{R,1}(v_0), p_{R,1}(v_1) \leq q_{R,1}(v) \quad (3.16)$$

for all $v \in T_{R,1}^\bullet(V)$.

Proof. The first part is a simple estimate; we have

$$\begin{aligned} p_{R,1}(v \otimes w) &= \sum_{k=0}^{\infty} p^k \left(\sum_{n+m=k} v_n \otimes w_m \right) k!^R \\ &\leq \sum_{k=0}^{\infty} \sum_{n+m=k} p^n(v_n) p^m(w_m) (n+m)!^R \\ &\leq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p^n(v_n) p^m(w_m) 2^{Rn} 2^{Rm} n!^R m!^R \end{aligned}$$

$$= (2^R p)_{R,1}(v)(2^R p)_{R,1}(w),$$

where we used $(n+m)! \leq 2^{n+m}n!m!$. Since with p also $2^R p$ is a continuous seminorm on V , the continuity of \otimes follows. The second and third part are standard, here we use the completed π -tensor product $V^{\otimes_\pi n}$ to achieve completeness at every fixed $n \in \mathbb{N}_0$. The fourth part is a consequence of the estimates (3.9) and (3.10). The last part follows from Lemma 3.1. \square

Remark 3.7 The case $R = 0$ gives a well-known topology on $T^\bullet(V)$ which becomes the *free locally multiplicatively convex* unital algebra generated by V as discussed e.g. by Cuntz in [11]. For $R > 0$ the completion $\hat{T}_{R,1}^\bullet(V)$ behaves differently: it does not even have an entire holomorphic calculus. To see this take the entire function

$$f_\epsilon(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!^\epsilon} \quad (3.17)$$

for a parameter $\epsilon > 0$. If $R > \epsilon$ then for every nonzero $v \in V$ the series $f_\epsilon(v)$ does not converge in $\hat{T}_{R,1}^\bullet(V)$. In particular, the tensor algebra $T_{R,1}^\bullet(V)$ can *not* be locally multiplicatively convex unless $R = 0$.

We have an analogous statement for the symmetric algebra. We equip $S^\bullet(V)$ with the induced topology from $T_{R,1}^\bullet(V)$ or $T_{R,\infty}^\bullet(V)$ and denote it by $S_{R,1}^\bullet(V)$ and $S_{R,\infty}^\bullet(V)$, respectively. From (3.5) we get immediately

$$p_{R,1}(\mathcal{S}(v)) \leq p_{R,1}(v), \quad (3.18)$$

which implies the continuity statement

$$p_{R,1}(vw) \leq (2^R p)_{R,1}(v)(2^R p)_{R,1}(w) \quad (3.19)$$

for all $v, w \in S^\bullet(V)$. This shows that $S_{R,1}^\bullet(V)$ becomes a locally convex algebra, too. Again, it will be locally multiplicatively convex only for $R = 0$ in which case it is the *free locally multiplicatively convex commutative* unital algebra generated by V . The completion of $S_{R,1}^\bullet(V)$ as well as the one of $S_{R,\infty}^\bullet(V)$ can be described analogously to (3.13) and (3.14). We also have the continuous inclusions

$$\hat{S}_{R',\infty}^\bullet(V) \longrightarrow \hat{S}_{R,1}^\bullet(V) \longrightarrow \hat{S}_{R,\infty}^\bullet(V) \quad (3.20)$$

for $R' > R$. Finally, we have the continuous projections and inclusions

$$S_{R,1}^\bullet(V) \longrightarrow S_\pi^n(V) \longrightarrow S_{R,1}^\bullet(V) \quad (3.21)$$

for all $n \in \mathbb{N}_0$. Thus it makes sense to speak of the n -th component v_n of a vector $v \in \hat{S}_{R,1}^\bullet(V)$ even after the completion. In fact, it is easy to see that the series of components

$$v = \sum_{n=0}^{\infty} v_n \quad (3.22)$$

converges to $v \in \hat{S}_{R,1}^\bullet(V)$, even absolutely.

To get rid of the somehow arbitrary parameter R we can pass to the projective limit $R \longrightarrow \infty$. The resulting locally convex algebras will be denoted by

$$\hat{T}_\infty^\bullet(V) = \text{proj lim}_{R \longrightarrow \infty} \hat{T}_{R,1}^\bullet(V) = \text{proj lim}_{R \longrightarrow \infty} \hat{T}_{R,\infty}^\bullet(V) \quad (3.23)$$

and by

$$\hat{S}_\infty^\bullet(V) = \text{proj lim}_{R \longrightarrow \infty} \hat{S}_{R,1}^\bullet(V) = \text{proj lim}_{R \longrightarrow \infty} \hat{S}_{R,\infty}^\bullet(V) \quad (3.24)$$

in the symmetric case. Note that thanks to the mutual inclusions (3.15) and (3.20) the difference between the ℓ^1 -version and the ℓ^∞ -version disappears. We have a more explicit description of $\hat{T}_\infty^\bullet(V)$ and $\hat{S}_\infty^\bullet(V)$ as consisting of those formal series $v = \sum_{n=0}^\infty v_n$ with $v_n \in V^{\hat{\otimes}_{\pi^n}}$ or $v_n \in \hat{S}_\pi^n(V)$, respectively, such that $p_{R,1}(v) < \infty$ for all $R \geq 0$ and for all continuous seminorms p on V . Equivalently, we can ask for $p_{R,\infty}(v) < \infty$. Note that these completions will be rather small as we require a rather strong decay of the coefficients v_n .

3.2 The Continuity of the Star Product

Let us now consider an even bilinear form $\Lambda: V \times V \longrightarrow \mathbb{K}$ which we require to be continuous. Thus there exists a continuous seminorm p on V such that

$$|\Lambda(v, w)| \leq p(v) p(w) \quad (3.25)$$

for all $v, w \in V$. Note that we require continuity and not just separate continuity. Note also, that if p satisfies (3.25) then we also have the estimates

$$|\Lambda_\pm(v, w)| \leq p(v) p(w), \quad (3.26)$$

showing the continuity of the antisymmetric and symmetric part of Λ . The continuity of Λ implies the continuity of the operator P_Λ when restricted to fixed symmetric degrees:

Lemma 3.8 *Let p be a continuous seminorm of V satisfying (3.25). Then for all $u \in S^n(V) \otimes S^m(V)$ one has*

$$(p^{n-1} \otimes p^{m-1})(P_\Lambda(u)) \leq nm p^{n+m}(u). \quad (3.27)$$

The same estimate holds for P_{Λ_\pm} .

Proof. We work on the whole tensor algebra first. Thus consider homogeneous vectors v_1, \dots, v_n and $w_1, \dots, w_m \in V$ and define $\tilde{P}_\Lambda: T^\bullet(V) \otimes T^\bullet(V) \longrightarrow T^\bullet(V) \otimes T^\bullet(V)$ by the linear extension of

$$\begin{aligned} & \tilde{P}_\Lambda(v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m) \\ &= \sum_{k=1}^n \sum_{\ell=1}^m (-1)^{v_k(v_{k+1}+\dots+v_n)} (-1)^{w_\ell(w_1+\dots+w_{\ell-1})} \Lambda(v_k, w_\ell) v_1 \otimes \dots \wedge^k \dots \otimes v_n \otimes w_1 \otimes \dots \wedge^\ell \dots \otimes w_m. \end{aligned}$$

Then we have $P_\Lambda \circ (\mathcal{S}_n \otimes \mathcal{S}_m) = (\mathcal{S}_{n-1} \otimes \mathcal{S}_{m-1}) \circ \tilde{P}_\Lambda$. For general tensors of arbitrary degree this yields

$$P_\Lambda \circ (\mathcal{S} \otimes \mathcal{S}) = (\mathcal{S} \otimes \mathcal{S}) \circ \tilde{P}_\Lambda. \quad (*)$$

For homogeneous vectors we get now the estimate

$$\begin{aligned} & (p^{n-1} \otimes p^{m-1}) \left(\tilde{P}_\Lambda(v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m) \right) \\ & \leq \sum_{k=1}^n \sum_{\ell=1}^m |\Lambda(v_k, w_\ell)| p(v_1) \dots \wedge^k \dots p(v_n) p(w_1) \dots \wedge^\ell \dots p(w_m) \\ & \leq nm p(v_1) \dots p(v_n) p(w_1) \dots p(w_m) \\ & = nm(p^n \otimes p^m)(v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m). \end{aligned}$$

By Lemma 3.2 we conclude that for all $u \in T^n(V) \otimes T^m(V)$ we have

$$(p^{n-1} \otimes p^{m-1}) \left(\tilde{P}_\Lambda(u) \right) \leq nm(p^n \otimes p^m)(u).$$

Finally, for $u \in S^n(V) \otimes S^m(V)$ we have $(\mathcal{S}_n \otimes \mathcal{S}_m)(u) = u$ and thus by $(*)$ and (3.5)

$$\begin{aligned}
(p^{n-1} \otimes p^{m-1})(P_\Lambda(u)) &= (p^{n-1} \otimes p^{m-1})(P_\Lambda(\mathcal{S}_n \otimes \mathcal{S}_m)(u)) \\
&= (p^{n-1} \otimes p^{m-1})((\mathcal{S}_{n-1} \otimes \mathcal{S}_{m-1})\tilde{P}_\Lambda(u)) \\
&\leq (p^{n-1} \otimes p^{m-1})(\tilde{P}_\Lambda(u)) \\
&\leq nm(p^n \otimes p^m)(u).
\end{aligned}$$

The last statement follows from (3.26). \square

In fact, this estimate just reflects the fact that P_Λ is a biderivation. If Λ is nontrivial then it can not be improved in general. It implies immediately the continuity of the Poisson bracket $\{\cdot, \cdot\}_\Lambda$:

Proposition 3.9 *Let Λ be continuous. Then the Poisson bracket $\{\cdot, \cdot\}_\Lambda$ is continuous on $S_{R,1}^\bullet(V)$ for every $R \geq 0$. More precisely, for $v, w \in S_{R,1}^\bullet(V)$ and any continuous seminorm p on V with (3.25) we have for all $\epsilon > 0$ a constant $c > 0$ such that*

$$p_{R,1}(\{v, w\}_\Lambda) \leq c(2^{R+\epsilon} p)_{R,1}(v)(2^{R+\epsilon} p)_{R,1}(w). \quad (3.28)$$

Proof. Let $v, w \in S_{R,1}^\bullet(V)$ with components $v = \sum_n v_n$ and $w = \sum_m w_m$ as usual. Then we have for a seminorm p satisfying (3.25)

$$\begin{aligned}
p_{R,1}(\{v, w\}_\Lambda) &= \sum_{k=0}^{\infty} p^k \left(\sum_{n+m-2=k} \{v_n, w_m\}_\Lambda \right) k!^R \\
&\leq \sum_{k=0}^{\infty} \sum_{n+m-2=k} p^{n+m-2} (2\mu \circ P_{\Lambda_-}(v_n \otimes w_m)) k!^R \\
&\leq \sum_{k=0}^{\infty} \sum_{n+m-2=k} 2nm p^n(v_n) p^m(w_m) (n+m-2)!^R \\
&\leq \sum_{n,m=0}^{\infty} 2nm 2^{(n+m)R} p^n(v_n) p^m(w_m) n!^R m!^R \\
&\leq c \sum_{n,m=0}^{\infty} 2^{(n+m)(R+\epsilon)} p^n(v_n) p^m(w_m) n!^R m!^R,
\end{aligned}$$

where we have used Lemma 3.8 for Λ_- , the standard estimate $(n+m)! \leq 2^{n+m} n! m!$ as well as a constant $c > 0$ with $2nm \leq c 2^{(n+m)\epsilon}$. \square

In this sense, $S_{R,1}^\bullet(V)$ becomes a *locally convex Poisson algebra* for every $R \geq 0$. In particular, the Poisson bracket extends to the completion $\hat{S}_{R,1}^\bullet(V)$ and still obeys the continuity estimate (3.28) as well as the algebraic properties of a Poisson bracket.

However, for the star product the situation is more complicated: first we note that thanks to Corollary 2.9 we can use the formal star product to get a well-defined *non-formal* star product $\star_{z\Lambda}$ by replacing ν with some real or complex number $z \in \mathbb{K}$, depending on our choice of the underlying field. We fix z in the following and consider the dependence of $\star_{z\Lambda}$ on z later in Subsection 3.4. The next lemma provides the key estimate for all continuity properties of $\star_{z\Lambda}$: the main point is that we can not stay with a single value of the parameter R .

Lemma 3.10 *Let $R \geq \frac{1}{2}$ and $\epsilon > 0$. Then there exists a constant $c_{z,\epsilon} > 0$ depending continuously on z and ϵ such that for all $a, b \in \mathbf{S}^\bullet(V)$ and all seminorms p with (3.25) we have*

$$p_{R,1}(a \star_{z\Lambda} b) \leq c_{z,\epsilon} p_{R+\epsilon,1}(a) p_{R+\epsilon,1}(b). \quad (3.29)$$

Proof. Let $a, b \in \mathbf{S}^\bullet(V)$ be given and denote by a_n, b_m their homogeneous parts with respect to the tensor degree as usual. Then we estimate

$$\begin{aligned} p_{R,1}(a \star_{z\Lambda} b) &\leq \sum_{k=0}^{\infty} \frac{|z|^k}{k!} p_{R,1}(\mu \circ P_\Lambda^k(a \otimes b)) \\ &= \sum_{k=0}^{\infty} \frac{|z|^k}{k!} p_{R,1} \left(\sum_{\ell=0}^{\infty} \sum_{\substack{n+m-2k=\ell \\ k \leq n,m}} \mu \circ P_\Lambda^k(a_n \otimes b_m) \right) \\ &= \sum_{k=0}^{\infty} \frac{|z|^k}{k!} \sum_{\ell=0}^{\infty} \ell!^R p^\ell \left(\sum_{\substack{n+m-2k=\ell \\ k \leq n,m}} \mu \circ P_\Lambda^k(a_n \otimes b_m) \right) \\ &\leq \sum_{k=0}^{\infty} \frac{|z|^k}{k!} \sum_{k \leq n,m} (n+m-2k)!^R p^{n+m-2k}(\mu \circ P_\Lambda^k(a_n \otimes b_m)) \\ &\stackrel{(a)}{\leq} \sum_{k=0}^{\infty} \sum_{k \leq n,m} (n+m-2k)!^R \frac{|z|^k}{k!} \frac{n!}{(n-k)!} \frac{m!}{(m-k)!} p^n(a_n) p^m(b_m) \\ &\stackrel{(b)}{\leq} \sum_{k=0}^{\infty} \sum_{k \leq n,m} \frac{|z|^k 2^{R(n+m-2k)}}{k!} \frac{n!}{(n-k)!^{1-R}} \frac{m!}{(m-k)!^{1-R}} p^n(a_n) p^m(b_m) \\ &= \sum_{k=0}^{\infty} \sum_{k \leq n,m} \frac{|z|^k 2^{R(n+m-2k)}}{k!} \frac{n!^{1-R-\epsilon}}{(n-k)!^{1-R}} \frac{m!^{1-R-\epsilon}}{(m-k)!^{1-R}} n!^{R+\epsilon} p^n(a_n) m!^{R+\epsilon} p^m(b_m) \\ &\stackrel{(c)}{\leq} \sum_{k=0}^{\infty} \sum_{k \leq n,m} |z|^k 2^{R(n+m-2k)} 2^{(1-R-\epsilon)n} 2^{(1-R-\epsilon)m} \frac{k!^{1-2R-2\epsilon}}{(n-k)!^\epsilon (m-k)!^\epsilon} n!^{R+\epsilon} p^n(a_n) m!^{R+\epsilon} p^m(b_m) \\ &\stackrel{(d)}{\leq} \sum_{k,n,m=0}^{\infty} |z|^k 2^{R(n+m-2k)} 2^{(1-R-\epsilon)n} 2^{(1-R-\epsilon)m} 2^{\frac{n\epsilon}{2}} 2^{\frac{m\epsilon}{2}} \frac{k!^{1-2R-\epsilon}}{n!^{\frac{\epsilon}{2}} m!^{\frac{\epsilon}{2}}} n!^{R+\epsilon} p^n(a_n) m!^{R+\epsilon} p^m(b_m) \\ &\stackrel{(e)}{\leq} \left(\sum_{k=0}^{\infty} \frac{|z|^k}{k!^\epsilon} \right) \left(\sum_{n=0}^{\infty} \frac{2^n}{n!^{\frac{\epsilon}{2}}} n!^{R+\epsilon} p^n(a_n) \right) \left(\sum_{m=0}^{\infty} \frac{2^m}{m!^{\frac{\epsilon}{2}}} m!^{R+\epsilon} p^m(b_m) \right). \end{aligned}$$

where in (a) we used k -times the estimate from Lemma 3.10, in (b) we used $(n+m-2k)! \leq 2^{n+m-2k} (n-k)!(m-k)!$, in (c) we used $n! \leq 2^n (n-k)!k!$ and $m! \leq 2^m (m-k)!k!$, in (d) we used $\frac{1}{(n-k)!} \leq 2^{\frac{n}{2}} \sqrt{\frac{k!}{n!}}$, and finally we used $R \geq \frac{1}{2}$ in (e). Now the first series yields an entire function of $|z|$ given by $f_\epsilon(|z|)$ from (3.17), and hence a continuous function of z . Moreover, in the second and the third series we can replace $\frac{2^n}{n!^{\frac{\epsilon}{2}}}$ and $\frac{2^m}{m!^{\frac{\epsilon}{2}}}$ again by $f_{\epsilon/2}(2)$. Clearly this depends continuously on $\epsilon > 0$. Combining these constants gives then the estimate (3.29). \square

Remark 3.11 Unlike for the Poisson bracket, a simple rescaling of the seminorm p to cp with some suitable constant $c > 0$ will not suffice in the estimate (3.29). The reason is that a polynomial in n of fixed degree like arising from P_Λ^k for a single k can be absorbed by c^n , but not the factorial $n!$ as occurring in $\star_{z\Lambda}$. Thus we need to pass from R to $R + \epsilon$ in order to achieve the estimate. This also motivates our choice of the weights $n!^R$ in the definitions of the seminorms $p_{R,1}$.

Remark 3.12 We also note that the limiting case $R = \frac{1}{2}$ is sharp in the following sense: consider the most simple nontrivial situation $V = \mathbb{R}^2$ with basis vectors q and p as well as the bilinear form $\Lambda_{\text{std}}(p, q) = 1$ and zero for the other combinations. The corresponding Poisson bracket is the canonical Poisson bracket and the star product is the *standard-ordered star product* if we take $z = \frac{\hbar}{i}$, see e.g. [24, Sect. 5.2.4]. Identifying elements in $S^\bullet(\mathbb{R}^2)$ with polynomials in q and p we have the more explicit formula

$$f \star_{\text{std}} g = \sum_{k=0}^{\infty} \frac{(-i\hbar)^k}{k!} \frac{\partial^k f}{\partial p^k} \frac{\partial^k g}{\partial q^k}. \quad (3.30)$$

Using again the function f_ϵ from (3.17) we see that $f_\epsilon(q)$ and $f_\epsilon(p)$ belong to $\hat{S}_{R,1}^\bullet(\mathbb{R}^2)$ as soon as $R < \epsilon$. However, for the star product we get (formally)

$$f_\epsilon(p) \star_{\text{std}} f_\epsilon(q) = \sum_{n,m,k \leq m,n} \frac{(-i\hbar)^k}{k!} \frac{n!^{1-\epsilon}}{(n-k)!} \frac{n!^{1-\epsilon}}{(n-k)!} q^{n-k} p^{n-k}. \quad (3.31)$$

Since the projection $\hat{S}_{R,1}^\bullet(V) \longrightarrow S_\pi^n(V)$ continuous for all $R \geq 0$ we consider the coefficient of (3.31) in $S^0(V)$ which is obtained for $n = m = k$, i.e.

$$\sum_{\ell=0}^{\infty} \frac{(-i\hbar)^\ell}{\ell!} \ell!^{1-\epsilon} \ell!^{1-\epsilon} = \sum_{\ell=0}^{\infty} (-i\hbar)^\ell \ell!^{1-2\epsilon}. \quad (3.32)$$

This clearly diverges for $\epsilon < \frac{1}{2}$ unless $\hbar = 0$. Thus for $R < \frac{1}{2}$ we can not expect a continuous star product.

3.3 The Weyl Algebra $\mathcal{W}_R(V, \star_{z\Lambda})$

As we have to move from R to $R + \epsilon$ in the fundamental estimate (3.29) we see that $\star_{z\Lambda}$ will *not* be a continuous product on $\hat{S}_{R,1}^\bullet(V)$. Thus we have to enhance our construction as follows:

Definition 3.13 (Weyl algebra) For $R \in \mathbb{R}$ we define the locally convex space

$$\mathcal{W}_R(V) = \text{proj} \lim_{\epsilon \rightarrow 0} \hat{S}_{R-\epsilon,1}^\bullet(V). \quad (3.33)$$

For $R > \frac{1}{2}$ we endow $\mathcal{W}_R(V)$ with the product $\star_{z\Lambda}$ and call the resulting algebra $\mathcal{W}_R(V, \star_{z\Lambda})$ the Weyl algebra.

More explicitly, this means that we use *all* seminorms $p_{R-\epsilon,1}$ for $\epsilon > 0$ and p a continuous seminorm on V . As a vector space we still have $\mathcal{W}_R(V) = S^\bullet(V)$ as before. The completion $\hat{\mathcal{W}}_R(V)$ will be given as those formal series $v = \sum_{n=0}^{\infty} v_n$ with $v_n \in \hat{S}_\pi^n(V)$ such that *all* seminorms $p_{R-\epsilon,1}(v)$ are finite for all $\epsilon > 0$ and all continuous seminorms p on V . Taking the projective limit $R \rightarrow \infty$ of the $\hat{S}_{R,1}^\bullet(V)$, i.e. the projective limit $S_\infty^\bullet(V)$ allows to define $\mathcal{W}_\infty(V) = S_\infty^\bullet(V)$. Most of the following statements will therefore also be available for the case $R = \infty$. However, we will not be too much interested in this case as the completion $\hat{\mathcal{W}}_\infty(V) = \hat{S}_\infty^\bullet(V)$ is rather small.

Remark 3.14 For later use, we can endow also the tensor algebra with the projective topology of all the seminorms $p_{R-\epsilon,1}$ for a fixed $R \in \mathbb{R}$ and all $\epsilon > 0$. The resulting tensor algebra will be denoted by $T_R^\bullet(V)$. Then $\mathcal{W}_R(V) \subseteq T_R^\bullet(V)$ is a closed subspace.

Remark 3.15 From Proposition 3.9 it follows immediately that the Poisson bracket $\{\cdot, \cdot\}_\Lambda$ is still continuous for this projective limit topology. Thus $\mathcal{W}_R(V)$ becomes a locally convex Poisson algebra.

We start now to collect some basic features of $\mathcal{W}_R(V)$. From Lemma 3.4 we get immediately the following statement:

Lemma 3.16 *Let $R' \geq R \geq 0$.*

- i.) For all $n \in \mathbb{N}_0$ the induced topology on $S^n(V) \subseteq \mathcal{W}_R(V)$ is the π -topology.*
- ii.) The projection and the inclusion maps*

$$\mathcal{W}_R(V) \longrightarrow S_\pi^n(V) \longrightarrow \mathcal{W}_R(V) \quad (3.34)$$

are continuous for all $n \in \mathbb{N}_0$.

- iii.) The inclusion map $\mathcal{W}_{R'}(V) \longrightarrow \mathcal{W}_R(V)$ is continuous.*
- iv.) The \mathbb{Z}_2 -grading is continuous for $\mathcal{W}_R(V)$.*

This lemma has the important consequence that also after completion of $\mathcal{W}_R(V)$ to $\hat{\mathcal{W}}_R(V)$ we can speak of the n -th component $a_n \in \hat{S}_\pi^n(V)$ of an element $a \in \hat{\mathcal{W}}_R(V)$ in a meaningful way. More precisely, a can be expressed as a convergent series in its components of fixed tensor degree:

Lemma 3.17 *Let $R \in \mathbb{R}$ and let $a \in \hat{\mathcal{W}}_R(V)$ with components $a_n \in \hat{S}_\pi^n(V)$ for $n \in \mathbb{N}_0$. Then*

$$a = \sum_{n=0}^{\infty} a_n \quad (3.35)$$

converges absolutely.

Proof. Identifying $a_n \in \hat{S}_\pi^n(V)$ with its image in $\hat{\mathcal{W}}_R(V)$ we get for every continuous seminorm p on V the equation

$$p_{R-\epsilon,1}(a) = \sum_{n=0}^{\infty} p^n(a_n) n!^{R-\epsilon} = \sum_{n=0}^{\infty} p_{R-\epsilon,1}(a_n),$$

from which the statement follows immediately. \square

In particular, the direct sum $\bigoplus_{n=0}^{\infty} \hat{S}_\pi^n(V)$ of the completed symmetric π -tensor powers of V is sequentially dense in $\hat{\mathcal{W}}_R(V)$.

Lemma 3.18 *Let $R \in \mathbb{R}$. Then the seminorms $p_{R-\epsilon,\infty}$ for $\epsilon > 0$ and p running through the continuous seminorms on V yield a defining system of seminorms for the topology of $\mathcal{W}_R(V)$.*

Proof. This follows from Lemma 3.4, *i.)* and *ii.)*. \square

The first main result is now that $\mathcal{W}_R(V, \star_{z\Lambda})$ is indeed a locally convex algebra provided R is suitably chosen:

Theorem 3.19 *Let $R > \frac{1}{2}$. The Weyl algebra $\mathcal{W}_R(V, \star_{z\Lambda})$ is a locally convex algebra. It is first countable iff V is first countable.*

Proof. The topology of $\mathcal{W}_R(V, \star_{z\Lambda})$ is defined by the collection of all seminorms $\{p_{R-\epsilon,1}\}_{\epsilon>0}$ with p running through the continuous seminorms of V . Clearly, it will be sufficient to consider only those ϵ where $R - \epsilon > \frac{1}{2}$ according to Lemma 3.4, *v.*), and those p with (3.25). For a resulting seminorm $p_{R-\epsilon,1}$ from this collection we find another $p_{R-\epsilon',1}$ where $0 < \epsilon' < \epsilon$, such that

$$p_{R-\epsilon,1}(a \star_{z\Lambda} b) \leq c_{z,\epsilon-\epsilon'} p_{R+\epsilon',1}(a) p_{R+\epsilon',1}(b)$$

for all $a, b \in \mathcal{W}_R(V)$ according to Lemma 3.10. This is the continuity of $\star_{z\Lambda}$. Note that we need $R > \frac{1}{2}$ for this argument. A locally convex space V is first countable iff we can choose a sequence of continuous seminorms $p^{(1)} \leq p^{(2)} \leq \dots$ such that for every other continuous seminorm q on V we have some $n \in \mathbb{N}$ with $q \leq p^{(n)}$. From Lemma 3.4 it is easy to see that the seminorms $(p^{(n)})_{R-\frac{1}{n},1}$ will determine the topology of $\mathcal{W}_R(V)$ in the same way. The converse is obvious from Lemma 3.16, *i.*). \square

Remark 3.20 Note that for a Banach space V a Fréchet algebra $\hat{\mathcal{W}}_R(V, \star_{z\Lambda})$ is the best we can hope for in general since the canonical commutation relations $[q, p]_{\star_{\text{std}}} = i\hbar \mathbb{1}$ can not be implemented in a Banach algebra for $\hbar \neq 0$. So if the even part of V and $\Lambda|_{V_0}$ are nontrivial, a Banach algebra structure is excluded since the standard-ordered star product from Remark 3.12 comprises always a subalgebra in this case. Even worse, in this case every submultiplicative seminorm is necessarily trivial: $\hat{\mathcal{W}}_R(V, \star_{z\Lambda})$ is very far from being locally multiplicatively convex.

As a first application we show that in the completion $\hat{\mathcal{W}}_R(V)$ we have exponentials of every vector in V , provided R is small enough:

Proposition 3.21 *Assume $V_0 \neq \{0\}$.*

- i.) One has $\exp(v) \in \hat{\mathcal{W}}_R(V)$ for every non-zero $v \in V$ iff $R \leq 1$.*
- ii.) Let $R \leq 1$ and $v \in V$. The map $\mathbb{K} \ni t \mapsto \exp(tv) \in \hat{\mathcal{W}}_R(V)$ is real-analytic in the case $\mathbb{K} = \mathbb{R}$ with radius of convergence ∞ and entire in the case $\mathbb{K} = \mathbb{C}$. The Taylor series converges absolutely.*

Proof. Let $v \in V_0$ be non-zero and choose a seminorm p with $p(v) > 0$. Then $p^n(v^n) = (p(v))^n$ since $v \otimes \dots \otimes v = v \cdots v$ in this case. The exponential series therefore gives

$$p_{R-\epsilon,1}(\exp(v)) = \sum_{n=0}^{\infty} \frac{p^n(v^n)}{n!} n!^{R-\epsilon} = \sum_{n=0}^{\infty} (p(v))^n n!^{R-\epsilon-1},$$

which converges for all $\epsilon > 0$ iff $R \leq 1$, showing the first part. The second part is clear from Lemma 3.17 since the homogeneous components of $\exp(tv)$ are given by $\frac{t^n v^n}{n!}$. \square

Note that in the case where $V_0 = \{0\}$ the exponential series of all elements $v \in V$ always converges since $\exp(v) = 1 + v$ for odd vectors $v \in V_1$. The proposition indicates that the limiting case of $R = 1$ deserves special attention in the following.

We conclude this subsection with a remark on yet another version of the Weyl algebra. Instead of fixing $R > \frac{1}{2}$ one can also perform an *inductive* limit $R \rightarrow \frac{1}{2}$ (from above) and define

$$\mathcal{W}_{\frac{1}{2}+}(V) = \text{ind} \lim_{R \rightarrow \frac{1}{2}} S_{R,1}^\bullet(V). \quad (3.36)$$

Then the star product $\star_{z\Lambda}$ is at least separately continuous on $\mathcal{W}_{\frac{1}{2}+}(V)$. However, the inductive limit does not behave as nicely as the projective one used for $\mathcal{W}_R(V)$. The reason is that $S_{R',1}^\bullet(V)$ is dense (equal) in the following $S_{R,1}^\bullet(V)$ for $R' \geq R$ but the topologies are different. Thus the inductive limit is *not* strict and it seems that not much can be said about $\mathcal{W}_{\frac{1}{2}+}(V)$.

3.4 Dependence on z

Up to now we have established the continuity of $\star_{z\Lambda}$ on $\mathcal{W}_R(V)$ and thus we can conclude that $\star_{z\Lambda}$ has a unique extension to a continuous product $\star_{z\Lambda}$ on the completion $\hat{\mathcal{W}}_R(V)$. We shall now re-interpret the proof of Lemma 3.10 to get the more specific statement that also the formula for $\star_{z\Lambda}$ stays valid:

Proposition 3.22 *Let $R > \frac{1}{2}$ and let $a, b \in \hat{\mathcal{W}}_R(V)$. Then*

$$a \star_{z\Lambda} b = \mu \circ e^{zP_\Lambda}(a \otimes b) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \mu \circ P_\Lambda^k(a \otimes b) \quad (3.37)$$

converges absolutely in $\hat{\mathcal{W}}_R(V)$.

Proof. We have to show that for all seminorms $p_{R-\epsilon,1}$ of the defining system of seminorms the series $\sum_{k=0}^{\infty} \frac{|z|^k}{k!} p_{R-\epsilon,1}(\mu \circ P_\Lambda^k(a \otimes b))$ converges. But this was exactly what we did in the proof of Lemma 3.10. \square

This proposition also allows us to discuss the dependence on the deformation parameter z : here we have the best possible scenario. In the real case, $z \in \mathbb{R}$, we have a real-analytic dependence on z in $a \star_{z\Lambda} b$ with an explicit Taylor expansion around $z = 0$ given by the absolutely convergent series (3.37). In the complex case, $z \in \mathbb{C}$, we have an entire dependence, again by (3.37). Note that it is important for such statements that the topology of $\mathcal{W}_R(V)$ is actually independent of z . Holomorphic deformations were introduced and studied in detail in [23], mainly in the context of Hopf algebra deformations.

Proposition 3.23 *Let $R > \frac{1}{2}$.*

i.) If $\mathbb{K} = \mathbb{R}$ then for every $a, b \in \hat{\mathcal{W}}_R(V)$ the map

$$\mathbb{R} \ni z \mapsto a \star_{z\Lambda} b \in \hat{\mathcal{W}}_R(V) \quad (3.38)$$

is real-analytic with Taylor expansion around $z = 0$ given by (3.37).

ii.) If $\mathbb{K} = \mathbb{C}$ then for every $a, b \in \hat{\mathcal{W}}_R(V)$ the map

$$\mathbb{C} \ni z \mapsto a \star_{z\Lambda} b \in \hat{\mathcal{W}}_R(V) \quad (3.39)$$

is holomorphic (even entire) with Taylor expansion around $z = 0$ given by (3.37). The collection of Weyl algebras $\{\hat{\mathcal{W}}_R(V, \star_{z\Lambda})\}_{z \in \mathbb{C}}$ provides a holomorphic (even entire) deformation of $\hat{\mathcal{W}}_R(V, \mu)$, where $\mu = \star_{z\Lambda}|_{z=0}$ is the symmetric tensor product.

3.5 Reality and the \ast -Involution

Up to now we treated the real and complex case on equal footing. However, for applications in physics one typically needs an additional structure, both for the classical Poisson algebra as well as for the quantum algebra: a reality structure in form of a \ast -involution.

The following two structures are well-known to be equivalent. We recall their relation in order to establish some notation: either we can start with a real vector space $V_{\mathbb{R}}$ and complexify it to $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$. This gives us an antilinear involutive automorphism of $V_{\mathbb{C}}$, the complex conjugation, denoted by $v_{\mathbb{R}} \otimes z \mapsto \overline{v_{\mathbb{R}} \otimes z} = v_{\mathbb{R}} \otimes \bar{z}$ for $v_{\mathbb{R}} \in V_{\mathbb{R}}$ and $z \in \mathbb{C}$. We can recover $V_{\mathbb{R}}$ as the real subspace of $V_{\mathbb{C}}$ of those vectors $v \in V_{\mathbb{C}}$ with $\bar{v} = v$. Or, equivalently, we can start with a complex vector space $V_{\mathbb{C}}$ and an antilinear involutive automorphism, still denoted by $v \mapsto \bar{v}$. Then $V_{\mathbb{C}} \cong V_{\mathbb{R}} \otimes \mathbb{C}$ with $V_{\mathbb{R}}$ consisting again of the real vectors in $V_{\mathbb{C}}$. In this situation the symmetric algebra $S^\bullet(V_{\mathbb{C}})$ is a \ast -algebra with respect to the complex conjugation, i.e. we have for homogeneous $a, b \in S^\bullet(V_{\mathbb{C}})$

$$\overline{ab} = (-1)^{ab} \bar{b} \bar{a} = \bar{a} \bar{b}, \quad (3.40)$$

where in the second equation we use the commutativity of the symmetric tensor product.

Remark 3.24 There are various reasons why this might not be what one really wants in the case of \mathbb{Z}_2 -graded algebras. Instead, an honest $*$ -involution *without* signs might be more desirable, i.e. $(ab)^* = b^*a^*$ for all a and b , no matter what parity they have. However, this requires some extra structure which we will not discuss in the sequel.

If in addition $V_{\mathbb{R}}$ is locally convex we can extend a continuous seminorm $p_{\mathbb{R}}$ on $V_{\mathbb{R}}$ to a seminorm $p_{\mathbb{C}}$ on $V_{\mathbb{C}}$ by setting $p_{\mathbb{C}}(v \otimes z) = |z| p_{\mathbb{R}}(v)$. This makes $V_{\mathbb{C}}$ a locally convex space such that the complex conjugation is continuous. In fact, $p_{\mathbb{C}}(\bar{v}) = p_{\mathbb{C}}(v)$ for the seminorms of the form $p_{\mathbb{C}}$. Conversely, if $V_{\mathbb{C}}$ is a locally convex complex vector space with a continuous complex conjugation then for every continuous seminorm q also $p(v) = \frac{1}{2}(q(v) + q(\bar{v}))$ is continuous, now satisfying $p(v) = p(\bar{v})$. Clearly, these seminorms still determine the topology of $V_{\mathbb{C}}$. Finally, for $p_{\mathbb{R}} = p|_{V_{\mathbb{R}}}$ we get $(p_{\mathbb{R}})_{\mathbb{C}} = p$. Thus also in the locally convex situation the two structures are equivalent.

Now let $V_{\mathbb{R}}$ be a real locally convex vector space and set $V = V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$, always endowed with the above locally convex topology and the canonical complex conjugation.

Lemma 3.25 *Let $R \in \mathbb{R}$ and let p be a continuous seminorm on V with $p(v) = p(\bar{v})$. Then*

$$p_{R,1}(a) = p_{R,1}(\bar{a}) \quad \text{and} \quad p_{R,\infty}(a) = p_{R,\infty}(\bar{a}) \quad (3.41)$$

for all $a \in S^{\bullet}(V)$. Thus the complex conjugation extends to a continuous antilinear involutive endomorphism of $S_{R,1}^{\bullet}(V)$ and $\mathcal{W}_R(V)$.

For a continuous \mathbb{C} -bilinear form $\Lambda: V \times V \rightarrow \mathbb{C}$ we define $\bar{\Lambda}: V \times V \rightarrow \mathbb{C}$ by

$$\bar{\Lambda}(v, w) = \overline{\Lambda(\bar{v}, \bar{w})} \quad (3.42)$$

as usual, and set $\text{Re}(\Lambda) = \frac{1}{2}(\Lambda + \bar{\Lambda})$ as well as $\text{Im}(\Lambda) = \frac{1}{2i}(\Lambda - \bar{\Lambda})$. Then $\bar{\Lambda}$, $\text{Re}(\Lambda)$, and $\text{Im}(\Lambda)$ are again continuous \mathbb{C} -bilinear forms.

In view of applications in quantum physics we rescale our deformation parameter $z \in \mathbb{C}$ to $z = \frac{i\hbar}{2}$ and consider only *real* (or even positive) values for \hbar . Thus the star product of a and $b \in \mathcal{W}_R(V)$ becomes

$$a \star_{\frac{i\hbar}{2}\Lambda} b = \mu \circ e^{\frac{i\hbar}{2}\Lambda}(a \otimes b). \quad (3.43)$$

The next result clarifies under which conditions the complex conjugation is a $*$ -involution for $\star_{\frac{i\hbar}{2}\Lambda}$:

Proposition 3.26 *Let $R > \frac{1}{2}$ and $\hbar \in \mathbb{R} \setminus \{0\}$. Then the following statements are equivalent:*

i.) The complex conjugation is a $$ -involution of $\hat{\mathcal{W}}_R(V, \star_{\frac{i\hbar}{2}\Lambda})$, i.e. we have for homogeneous $a, b \in \hat{\mathcal{W}}_R(V)$*

$$\overline{a \star_{\frac{i\hbar}{2}\Lambda} b} = (-1)^{ab} \bar{b} \star_{\frac{i\hbar}{2}\Lambda} \bar{a}. \quad (3.44)$$

ii.) $\bar{\Lambda}_+ = -\Lambda_+$ and $\bar{\Lambda}_- = \Lambda_-$.

iii.) $\bar{\Lambda}_+ = -\Lambda_+$ and $\hat{\mathcal{W}}_R(V)$ is a Poisson $$ -algebra in the sense that for all $a, b \in \hat{\mathcal{W}}_R(V)$*

$$\overline{\{a, b\}_{\Lambda}} = \{\bar{a}, \bar{b}\}_{\Lambda}. \quad (3.45)$$

Proof. First we note that by continuity it suffices to work on $\mathcal{W}_R(V)$ instead of the completion. Suppose *i.)* and consider homogeneous $v, w \in V$. Then

$$\begin{aligned} 0 &= \overline{v \star_{\frac{i\hbar}{2}\Lambda} w} - (-1)^{vw} \bar{w} \star_{\frac{i\hbar}{2}\Lambda} \bar{v} \\ &= -\frac{i\hbar}{2} (\bar{\Lambda}(\bar{v}, \bar{w}) + (-1)^{vw} \Lambda(\bar{w}, \bar{v})) \end{aligned}$$

$$= -\frac{i\hbar}{2}(\overline{\Lambda}_+(\overline{v}, \overline{w}) + \Lambda_+(\overline{w}, \overline{v}) + \overline{\Lambda}_-(\overline{v}, \overline{w}) - \Lambda_-(\overline{w}, \overline{v})),$$

since the star product gives only zeroth and first order terms and Λ_+ is symmetric while Λ_- is antisymmetric. Now $\overline{\Lambda}_+ + \Lambda_+$ is still symmetric and $\overline{\Lambda}_- - \Lambda_-$ is still antisymmetric. Hence their contributions have to vanish separately which implies *ii.*). Next, assume *ii.*). Then

$$\overline{P_{\Lambda_{\pm}}(a \otimes b)} = \mp P_{\Lambda_{\pm}}(\overline{a} \otimes \overline{b}) \quad (*)$$

follows immediately. For the symmetric tensor product μ we $\overline{\mu(a \otimes b)} = \mu(\overline{a} \otimes \overline{b})$ which combines to give *iii.*) at once. Conversely, *iii.*) implies *ii.*) by evaluating on $v, w \in V$. Finally, assume *ii.*). In general the (anti-) symmetry of Λ_{\pm} implies

$$P_{\Lambda_{\pm}} = \pm \tau \circ P_{\Lambda_{\pm}} \circ \tau, \quad (**)$$

as this follows either by a direct computation or by verifying the Leibniz rules for $\tau \circ P_{\Lambda_{\pm}} \circ \tau$ and then applying the uniqueness result from Lemma 2.2. Combining now (*) and (**) with the commutativity $\mu = \mu \circ \tau$ of the symmetric tensor product gives *i.*) by a computation analogously to [24, Prop. 5.2.19]. \square

Thus we need a real Poisson bracket to start the deformation and an imaginary symmetric part Λ_+ in the star product $\star_{\frac{i\hbar}{2}\Lambda}$ to have the complex conjugation as $*$ -involution. In this case $\hat{\mathcal{W}}_R(V, \star_{\frac{i\hbar}{2}\Lambda})$ is a locally convex $*$ -algebra.

4 Bases and Nuclearity

In this section we collect some additional properties of the Weyl algebra $\mathcal{W}_R(V, \star_{z\Lambda})$ which are inherited from V .

4.1 Absolute Schauder Bases

Suppose that V has an absolute Schauder basis, i.e. there exists a linearly independent set $\{e_i\}_{i \in I}$ of vectors in V together with continuous coefficient functionals $\{\varphi^i\}_{i \in I}$ in V' such that $\varphi^i(e_j) = \delta_j^i$ for $i, j \in I$ and

$$v = \sum_{i \in I} \varphi^i(v) e_i \quad (4.1)$$

converges. More precisely, for an *absolute* Schauder basis one requires that for all continuous seminorms p on V there exists a continuous seminorm q such that

$$\sum_{i \in I} |\varphi^i(v)| p(e_i) \leq q(v) \quad (4.2)$$

for all $v \in V$, i.e. the series in (4.1) converges absolutely for all continuous seminorms and can be estimated by a continuous seminorm. In particular, at most countably many contributions $\varphi^i(v) p(e_i)$ can be different from 0 for a given $v \in V$. Typically, I will be countable itself. In the following we assume to have such an absolute Schauder basis $\{e_i\}_{i \in I}$ for V with coefficient functionals $\{\varphi^i\}_{i \in I}$.

The projective topology on $V^{\otimes n}$ is known to be compatible with absolute Schauder bases, More precisely, we have the following folklore lemma:

Lemma 4.1 *The vectors $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{i_1, \dots, i_n \in I}$ form an absolute Schauder basis for $V^{\otimes \pi n}$ with coefficient functionals $\{\varphi^{i_1} \otimes \cdots \otimes \varphi^{i_n}\}_{i_1, \dots, i_n \in I}$. If p and q are continuous seminorms on V with (4.2) then one has for all $v \in V^{\otimes \pi n}$*

$$\sum_{i_1, \dots, i_n \in I} |(\varphi^{i_1} \otimes \cdots \otimes \varphi^{i_n})(v)| p^n(e_{i_1} \otimes \cdots \otimes e_{i_n}) \leq q^n(v). \quad (4.3)$$

In a next step we consider the whole tensor algebra $T_{R,1}^\bullet(V)$ endowed with the topology from Definition 3.5 for some fixed $R \in \mathbb{R}$. We claim that the collection $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{n \in \mathbb{N}_0, i_1, \dots, i_n \in I}$ provides an absolute Schauder basis for $T_{R,1}^\bullet(V)$, where the coefficient functionals are $\{\varphi^{i_1} \otimes \cdots \otimes \varphi^{i_n}\}_{n \in \mathbb{N}_0, i_1, \dots, i_n \in I}$. Here for $n = 0$ we take the standard basis vector $1 \in \mathbb{K}$ with the corresponding coefficient functional. First we note that the linear functionals $\varphi^{i_1} \otimes \cdots \otimes \varphi^{i_n}$ are continuous on $T_{R,1}^\bullet(V)$ when they are extended by 0 to the tensor degrees different from n . Moreover, we have

$$p_{R,1}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = p^n(e_{i_1} \otimes \cdots \otimes e_{i_n})n!^R \quad (4.4)$$

by Lemma 3.4, *iii.*). This results in the following statement:

Lemma 4.2 *The vectors $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{n \in \mathbb{N}_0, i_1, \dots, i_n \in I}$ form an absolute Schauder basis for $T_{R,1}^\bullet(V)$ with coefficient functionals $\{\varphi^{i_1} \otimes \cdots \otimes \varphi^{i_n}\}_{n \in \mathbb{N}_0, i_1, \dots, i_n \in I}$. If p and q are continuous seminorms on V with (4.2) then one has for all $v \in T_{R,1}^\bullet(V)$*

$$\sum_{n=0}^{\infty} \sum_{i_1, \dots, i_n \in I} |(\varphi^{i_1} \otimes \cdots \otimes \varphi^{i_n})(v)| p_{R,1}(e_{i_1} \otimes \cdots \otimes e_{i_n}) \leq q_{R,1}(v). \quad (4.5)$$

Remark 4.3 We note that an *absolute* Schauder basis stays an absolute Schauder basis after completion, see [20, Prop. 14.7.7].

The absolute Schauder basis descends now to the symmetric algebra by symmetrizing. If $v \in S^\bullet(V)$ then we have $v = S v$ with the continuous symmetrization map from (3.18). Applying S twice, this shows the convergence

$$v = \sum_{n=0}^{\infty} \sum_{i_1, \dots, i_n \in I} (\varphi^{i_1} \cdots \varphi^{i_n})(v) e_{i_1} \cdots e_{i_n} \quad (4.6)$$

for all $v \in S_{R,1}^\bullet(V)$, where $\varphi^{i_1} \cdots \varphi^{i_n} = (\varphi^{i_1} \otimes \cdots \otimes \varphi^{i_n}) \circ S$. Moreover, using again $v = S v$ we get from the estimate (3.18) and (4.5) the estimate

$$\sum_{n=0}^{\infty} \sum_{i_1, \dots, i_n \in I} |(\varphi^{i_1} \cdots \varphi^{i_n})(v)| p_{R,1}(e_{i_1} \cdots e_{i_n}) \leq q_{R,1}(v). \quad (4.7)$$

So we have all we need for an absolute Schauder basis *except* that the symmetrizations $e_{i_1} \cdots e_{i_n}$ will no longer be linearly independent: some of them will be zero if they contain twice the same odd vector and some of them will differ by signs. So we only have to single out a maximal linearly independent subset, the choice of which might be personal taste:

Proposition 4.4 *Let $R \in \mathbb{R}$ and let $\{e_i\}_{i \in I}$ be an absolute Schauder basis of V of homogeneous vectors with coefficient functionals $\{\varphi^i\}_{i \in I}$. Then any choice of a maximal linearly independent subset of $\{e_{i_1} \cdots e_{i_n}\}_{n \in \mathbb{N}_0, i_1, \dots, i_n \in I}$ will give an absolute Schauder basis of $S_{R,1}^\bullet(V)$ and of $\mathcal{W}_R(V)$ with coefficient functionals given by the corresponding subset of $\{\varphi^{i_1} \cdots \varphi^{i_n}\}_{n \in \mathbb{N}_0, i_1, \dots, i_n \in I}$. One has the estimate (4.7) whenever p and q satisfy (4.2).*

Having an absolute Schauder basis is a very strong property for a locally convex space. In fact, they are completely known: after completion one obtains a Köthe (sequence) space where the index set for the “sequences” is I , see e.g. [20, Sect. 1.7E and Sect. 14.7] for a detailed description. More precisely, the Köthe matrix K_V of V is obtained from $K_V = (\lambda_{i,p})$ with $\lambda_{i,p} = p(e_i)$ where $i \in I$ and p ranges over a defining system of continuous seminorms of V . Thus the corresponding Köthe matrix of the tensor algebra $T_{R,1}^\bullet(V)$ is given by $K_{T_{R,1}^\bullet(V)} = (\lambda_{(n,i_1, \dots, i_n), p})$ with

$$\lambda_{(n,i_1, \dots, i_n), p} = n!^R \lambda_{i_1, p} \cdots \lambda_{i_n, p}. \quad (4.8)$$

Thus we have an *explicit* description in terms of the Köthe matrix of V . Analogously, one can proceed for $S_{R,1}^\bullet(V)$ and the Weyl algebra $\mathcal{W}_R(V)$. For Köthe spaces many properties are (easily) encoded in their Köthe matrix, so we see here that the appearance of $n!^R$ will play a prominent role when passing from V to $T_{R,1}^\bullet(V)$ or $\mathcal{W}_R(V)$.

4.2 Nuclearity of $\mathcal{W}_R(V)$

Let us now discuss nuclearity properties of the Weyl algebra $\mathcal{W}_R(V)$ originating from those of V : since $V \subseteq \mathcal{W}_R(V)$ is a closed subspace inheriting the original topology from the one of $\mathcal{W}_R(V)$, we see that nuclearity of $\mathcal{W}_R(V)$ implies the nuclearity of V . The aim of this subsection is to show the converse.

To this end, it will be convenient to work with the tensor algebra $T_{R,1}^\bullet(V)$ instead of the symmetric algebra $S_{R,1}^\bullet(V)$ and $\mathcal{W}_R(V)$ since we do not have to take care of the combinatorics of symmetrization.

Let $U \subseteq V$ be a subspace and denote by $\langle U \rangle \subseteq T^\bullet(V)$ the two-sided ideal generated by U . Then the quotient algebra $T^\bullet(V)/\langle U \rangle$ is still \mathbb{Z} -graded by the tensor degree since U has homogeneous generators of tensor degree one. The map

$$\iota: T^\bullet(V)/\langle U \rangle \longrightarrow T^\bullet(V/U) \quad (4.9)$$

determined by $\iota([v_1 \otimes \cdots \otimes v_n]) = [v_1] \otimes \cdots \otimes [v_n]$ turns out to be an isomorphism of graded algebras. We shall now show that ι also respects the seminorms $p_{R,1}$. First recall that for a seminorm p on V one defines a seminorm $[p]$ on V/U by

$$[p]([v]) = \inf\{p(v + u) \mid u \in U\}. \quad (4.10)$$

Then the locally convex quotient topology on V/U is obtained from all the seminorms $[p]$ where p runs through all the continuous seminorms of V .

Lemma 4.5 *Let $U \subseteq V$ be a subspace and let p be a seminorm on V . Then for all $R \in \mathbb{R}$ one has*

$$[p_{R,1}] = [p]_{R,1} \circ \iota. \quad (4.11)$$

Proof. Let $\langle U \rangle_n = \sum_{\ell=1}^n V \otimes \cdots \otimes U \otimes \cdots \otimes V \subseteq V^{\otimes n}$ with U being at the ℓ -th position. This is the n -th homogeneous part of $\langle U \rangle$. Then $V^{\otimes n}/\langle U \rangle_n$ gives the n -th homogeneous part of the graded algebra $T^\bullet(V)/\langle U \rangle$. For a seminorm p on V and the induced isomorphism ι restricted to $V^{\otimes n}/\langle U \rangle_n$ a simple argument shows

$$[p^n] = [p]^n \circ \iota.$$

From this, (4.11) follows at once. □

This simple lemma has an important consequence which we formulate in two ways:

Corollary 4.6 *Let $U \subseteq V$ be a subspace and $R \in \mathbb{R}$.*

i.) The isomorphism (4.9) induces an isomorphism

$$\iota: T_{R,1}^\bullet(V)/\langle U \rangle \longrightarrow T_{R,1}^\bullet(V/U) \quad (4.12)$$

of locally convex algebras if the left hand side as well as V/U carry the locally convex quotient topologies.

ii.) The isomorphism (4.9) induces an isomorphism

$$\iota: T_R^\bullet(V)/\langle U \rangle \longrightarrow T_R^\bullet(V/U) \quad (4.13)$$

of locally convex algebras.

Proof. For the second part, we note that the seminorms $[p_{R-\epsilon,1}]$ and $[p]_{R-\epsilon,1}$ for $\epsilon > 0$ and p a continuous seminorm on V constitute a defining system of seminorms for the projective limit topologies. \square

Let us now assume that V is nuclear. There are many equivalent ways to characterize nuclearity, see e.g. [20, Chap. 21], we shall use the following very basic one: for a given continuous seminorm p on V we consider $V/\ker p$ with the quotient seminorm $[p]$. This is now a normed space as we have divided by $\ker p$. Thus we can complete $V/\ker p$ to a Banach space denoted by V_p . Then V is called nuclear if for every continuous seminorm p there is another continuous seminorm $q \geq p$ such that the canonical map $\iota_{qp}: V_q \rightarrow V_p$ is a nuclear map. This means that there are vectors $e_i \in V_p$ and continuous linear functionals $\epsilon^i \in V'_q$ such that

$$\iota_{qp}(v) = \sum_{i=1}^{\infty} \epsilon^i(v) e_i \quad \text{with} \quad \sum_{i=1}^{\infty} \|\epsilon^i\|_q \|e_i\|_p < \infty, \quad (4.14)$$

where we use the notation $\|\cdot\|_p = [p]$ for the Banach norms on V_p and

$$\|\epsilon^i\|_q = \sup_{v \neq 0} \frac{|\epsilon^i(v)|}{\|v\|_q} \quad (4.15)$$

denotes the functional norm as usual. The following lemma is well-known:

Lemma 4.7 *Let $(V, \|\cdot\|)$ be a Banach space and let $\varphi_1, \dots, \varphi_n \in V'$. Then $\varphi_1 \otimes \dots \otimes \varphi_n \in (V^{\otimes n})'$ with*

$$\|\varphi_1 \otimes \dots \otimes \varphi_n\| = \|\varphi_1\| \dots \|\varphi_n\|, \quad (4.16)$$

where on $V^{\otimes n}$ we use the norm $\|\cdot\| \otimes \dots \otimes \|\cdot\|$ as usual.

The next lemma shows how $\ker p \subseteq V$ is related to $\ker p_{R,1} \subseteq T_{R,1}^\bullet(V)$.

Lemma 4.8 *Let $R \in \mathbb{R}$ and let p be a continuous seminorm on V . Then*

$$\langle \ker p \rangle = \ker p_{R,1}. \quad (4.17)$$

Proof. Since $[p]$ is a norm on $V/\ker p$, also $[p]^n$ is a norm on $(V/\ker p)^{\otimes n}$. It follows that $[p]_{R,1}$ is a norm on $T^\bullet(V/\ker p)$ as well, implying that $[p_{R,1}]$ is a norm on $T(V)/\langle \ker p \rangle$ according to Lemma 4.5. Thus $[v] \in \ker p_{R,1}$ iff $[p_{R,1}](v) = 0$ iff $[v] = 0$ iff $v \in \langle \ker p \rangle$. \square

The following lemma is the key to understand nuclearity:

Lemma 4.9 *Let $p \leq q$ be continuous seminorms such that the canonical map $\iota_{qp}: V_q \rightarrow V_p$ has a nuclear representation*

$$\iota_{qp} = \sum_{i=1}^{\infty} \epsilon^i \otimes e_i \quad (4.18)$$

with $\epsilon^i \in V'_q$ and $e_i \in V_p$. Then the canonical map

$$\iota_{q_{R',1} p_{R,1}}: T_{R',1}^\bullet(V)_{q_{R',1}} \rightarrow T_{R,1}^\bullet(V)_{p_{R,1}} \quad (4.19)$$

has the nuclear representation

$$\iota_{q_{R',1} p_{R,1}} = \sum_{n=0}^{\infty} \sum_{i_1, \dots, i_n=1}^{\infty} (\epsilon^{i_1} \otimes \dots \otimes \epsilon^{i_n}) \otimes (e_{i_1} \otimes \dots \otimes e_{i_n}) \quad (4.20)$$

whenever $R' > R$.

Proof. First we note that $(\mathbf{T}_{R',1}^\bullet(V))_{q_{R',1}}$ is the Banach space completion of $\mathbf{T}_{R',1}^\bullet(V)/\ker q_{R',1} \cong \mathbf{T}_{R',1}^\bullet(V/\ker q)$ with respect to the norm $[q_{R',1}] = [q]_{R',1} \circ \iota$, according to Lemma 4.5, and analogously for $(\mathbf{T}_{R,1}^\bullet(V))_{p_{R,1}}$. In this sense we have $e_{i_1} \otimes \cdots \otimes e_{i_n} \in (\mathbf{T}_{R,1}^\bullet(V))_{p_{R,1}}$ as well as $\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_n} \in ((\mathbf{T}_{R',1}^\bullet(V))_{q_{R',1}})'$. For the norms of these vectors and linear functionals we have

$$p_{R,1}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = n!^R [p]^n(e_{i_1} \otimes \cdots \otimes e_{i_n}) = n!^R \|e_{i_1}\|_p \cdots \|e_{i_n}\|_p$$

and

$$\|\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_n}\|_{q_{R',1}} = \frac{1}{n!^{R'}} \|\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_n}\|_{q^n} = \frac{1}{n!^{R'}} \|\epsilon^{i_1}\|_q \cdots \|\epsilon^{i_n}\|_q.$$

Note that due to dualizing, the prefactor $n!^{R'}$ appears now in the denominator. Denote by

$$c = \sum_{i=1}^{\infty} \|\epsilon^i\|_q \|e_i\|_p$$

the numerical value of the convergence condition in (4.14). Then we have

$$\sum_{n=0}^{\infty} \sum_{i_1, \dots, i_n=0}^{\infty} \|\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_n}\|_{q_{R',1}} p_{R,1}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \sum_{n=0}^{\infty} \frac{c^n}{n!^{R'-R}} < \infty$$

whenever $R' > R$. Finally, it is clear that (4.20) holds before the completion and thus also afterwards by continuity. \square

Theorem 4.10 *Let $R \in \mathbb{R}$. Then the following statements are equivalent:*

- i.) V is nuclear.
- ii.) $\mathbf{T}_R^\bullet(V)$ is nuclear.
- iii.) $\mathcal{W}_R(V)$ is nuclear.

Proof. We consider the tensor algebra $\mathbf{T}_R^\bullet(V)$ equipped with the projective limit topology of all the $\mathbf{T}_{R-\epsilon,1}^\bullet(V)$ with $\epsilon > 0$ according to Remark 3.14. Then $\mathcal{W}_R(V)$ is a closed subspace of $\mathbf{T}_R^\bullet(V)$ and V is a closed subspace of $\mathcal{W}_R(V)$. Hence it will suffice to show that $\mathbf{T}_R^\bullet(V)$ is nuclear whenever V is nuclear. The projective limit topology is determined by all the seminorms $p_{R-\epsilon,1}$ with $\epsilon > 0$ and p being a continuous seminorm on V . For a given p , we find a continuous seminorm q on V such that $\iota_{q,p}: V_q \rightarrow V_p$ has a nuclear representation. But then Lemma 4.9 shows that $q_{R-\epsilon',1}$ will do the job for $p_{R-\epsilon,1}$ as soon as we have $0 < \epsilon' < \epsilon$. Hence $\mathbf{T}_R^\bullet(V)$ is nuclear. \square

Remark 4.11 (Strong nuclearity) The same argument also applies to strong nuclearity where the sequence $\|\epsilon^i\|_q \|e_i\|_p$ is required to be in ℓ^p for all $p > 0$ instead of just being ℓ^1 . Clearly, an arbitrary positive power of a factorial in the denominator can handle this situation as well. Hence $\mathcal{W}_R(V)$ is strongly nuclear iff V is strongly nuclear.

Example 4.12 For V finite-dimensional we get a strongly nuclear Weyl algebra $\mathcal{W}_R(V)$. Here we can either use the above theorem since V is strongly nuclear, or we can rely on the explicit description of V and hence of $\mathcal{W}_R(V)$ as a Köthe space: since for a finite-dimensional vector space it suffices to take a single norm, the Köthe matrix is finite and hence its entries are bounded, say by $c > 0$. Thus the Köthe matrix (4.8) has entries bounded by $c^n n!^R$. To this result one can apply the Grothendieck-Pietsch criterion and conclude strong nuclearity directly, see [20, Prop. 21.8.2].

5 Symmetries and Equivalences

In this section we discuss how the algebraic symmetries and equivalences translate into our locally convex framework.

5.1 Functoriality of $\hat{\mathcal{W}}_R(V, \star_{z\Lambda})$

Suppose that V and W are two \mathbb{Z}_2 -graded locally convex Hausdorff spaces and let Λ_V and Λ_W be continuous even bilinear forms on V and W , respectively. We want to extend the functoriality statement from Proposition 2.15. The following estimates are obvious:

Lemma 5.1 *Let $A: V \rightarrow W$ be an even linear map and let p and q be seminorms on V and W such that $q(A(v)) \leq p(v)$ for all $v \in V$. Then*

$$q_{R,1}(A(v)) \leq p_{R,1}(v) \quad \text{and} \quad q_{R,\infty}(A(v)) \leq p_{R,\infty}(v) \quad (5.1)$$

for all $v \in T^\bullet(V)$.

Proof. We clearly have $q^n(A^{\otimes n}(v)) \leq p^n(v)$ for all $v \in T^n(V)$. From this, the estimates are clear by the definition of the seminorms $p_{R,1}$ and $p_{R,\infty}$. \square

Proposition 5.2 *Let $R > \frac{1}{2}$ and $z \in \mathbb{C}$. Then the Weyl algebra $\mathcal{W}_R(V, \star_{z\Lambda})$ as well as its completion $\hat{\mathcal{W}}_R(V, \star_{z\Lambda})$ depend functorially on (V, Λ) with respect to continuous Poisson maps.*

In particular, the continuous Poisson automorphisms in $\text{Aut}(V, \Lambda)$ act on the Weyl algebra $\mathcal{W}_R(V, \star_{z\Lambda})$ as well as on its completion $\hat{\mathcal{W}}_R(V, \star_{z\Lambda})$ by continuous automorphisms.

5.2 Translational Invariance

Let us now investigate the action of the translations by linear forms on V as done algebraically in (2.24). We discuss the continuity of the translations τ_φ^* in two ways: first directly for a general even continuous $\varphi \in V'$ and second for a φ in the image of \sharp from (2.27): since Λ is continuous, an element $\varphi \in \text{im } \sharp \subseteq V^*$ is clearly continuous as well. In this more special situation we show the continuity of τ_φ^* by showing some much stronger statement, namely that τ_φ^* is an inner automorphism.

We start with the following basic estimate for the continuity of the translation operators τ_φ^* :

Lemma 5.3 *Let $\varphi \in V'$ be even and let p be a continuous seminorm on V such that $|\varphi(v)| \leq p(v)$ for all $v \in V$. Then for $R \geq 0$ we have for all $v \in T_{R,1}^\bullet(V)$*

$$p_{R,1}(\tau_\varphi^* v) \leq (2p)_{R,1}(v). \quad (5.2)$$

Proof. We write $v = \sum_{n=0}^\infty v_n \in T^\bullet(V)$ with its homogeneous components $v_n \in V^{\otimes n}$, all of which are zero except finitely many. Moreover, we write

$$v_n = \sum_i v_i^{(1)} \otimes \cdots \otimes v_i^{(n)} \quad (*)$$

as usual. Then the homomorphism property of τ_φ^* gives

$$\tau_\varphi^* v = \sum_{n=0}^\infty \sum_i \left(v_i^{(1)} + \varphi(v_i^{(1)}) \mathbb{1} \right) \otimes \cdots \otimes \left(v_i^{(n)} + \varphi(v_i^{(n)}) \mathbb{1} \right).$$

For every n we get now various contributions in all tensor degrees $k \leq n$. The contributions in the tensor degree k consists linear combinations of a choice of k vectors among the $v_i^{(1)}, \dots, v_i^{(n)}$, taking

their tensor product, applying φ to the remaining $n - k$ vectors, and multiplying everything together in the end. For a fixed index i there are $\binom{n}{k}$ possibilities to distribute $n - k$ copies of φ to the n vectors $v_i^{(1)}, \dots, v_i^{(n)}$. Finally, using the estimate $|\varphi(w)| \leq p(w)$ for all $w \in V$ we obtain that the contributions to $p_{R,1}$ from these terms can be estimated by

$$p_{R,1} \left(\left(v_i^{(1)} + \varphi(v_i^{(1)}) \mathbb{1} \right) \otimes \dots \otimes \left(v_i^{(n)} + \varphi(v_i^{(n)}) \mathbb{1} \right) \right) \leq \sum_{k=0}^n \binom{n}{k} k!^R p(v_i^{(1)}) \dots p(v_i^{(n)}).$$

In total, we get the estimate

$$p_{R,1}(\tau_\varphi^* v) \leq \sum_{n=0}^{\infty} \sum_i \sum_{k=0}^n \binom{n}{k} k!^R p(v_i^{(1)}) \dots p(v_i^{(n)}) \leq \sum_{n=0}^{\infty} \sum_i 2^n n!^R p(v_i^{(1)}) \dots p(v_i^{(n)}).$$

Since the decomposition $(*)$ was arbitrary, we can take the infimum over all such decompositions resulting in (5.2). \square

From this estimate we get immediately the following continuity statements:

Proposition 5.4 *Let $\varphi \in V'$ be an even continuous linear functional and let $R \geq 0$.*

- i.) *The algebra automorphism $\tau_\varphi^*: \mathbf{T}_{R,1}^\bullet(V) \longrightarrow \mathbf{T}_{R,1}^\bullet(V)$ is continuous.*
- ii.) *The Poisson algebra automorphism $\tau_\varphi^*: (\mathbf{S}_{R,1}^\bullet(V), \{\cdot, \cdot\}_\Lambda) \longrightarrow (\mathbf{S}_{R,1}^\bullet(V), \{\cdot, \cdot\}_\Lambda)$ is continuous.*
- iii.) *The algebra automorphism $\tau_\varphi^*: \mathbf{T}_R^\bullet(V) \longrightarrow \mathbf{T}_R^\bullet(V)$ is continuous.*
- iv.) *For $R > \frac{1}{2}$, the algebra automorphism $\tau_\varphi^*: \mathcal{W}_R(V, \star_{z\Lambda}) \longrightarrow \mathcal{W}_R(V, \star_{z\Lambda})$ is continuous.*

In particular, τ_φ^* extends in all four cases to the corresponding completions and yields a continuous automorphism for the completions, too.

In a next step we want to understand which of the τ_φ^* are inner automorphisms. Heuristically, this is a well-known statement: if the linear functional φ is in the image of \sharp then τ_φ^* is inner via the star exponential of a pre-image of φ with respect to \sharp . Also the heuristic formula for the star-exponential is folklore. Our main point here is that we have an analytical framework where the star-exponential actually makes sense: this is in so far nontrivial as we know that the canonical commutation relations do not allow for a general entire calculus, see the discussion in Remark 3.7 and Remark 3.20. Thus the existence of an exponential has to be shown by hand.

In the following it will be crucial to have $R \leq 1$ in view of Proposition 3.21. We start with some basic properties of the exponential series:

Lemma 5.5 *Let $R \leq 1$ and $w \in V_0$ be an even vector.*

- i.) *For all $v \in V$ we have*

$$\exp(w) \star_{z\Lambda} v = \exp(w)(v + z\Lambda(w, v)), \quad (5.3)$$

$$v \star_{z\Lambda} \exp(w) = \exp(w)(v + z\Lambda(v, w)). \quad (5.4)$$

- ii.) *For all $t \in \mathbb{K}$ one has*

$$\frac{d}{dt} e^{tw + \frac{t^2 z}{2} \Lambda(w, w) \mathbb{1}} = e^{tw + \frac{t^2 z}{2} \Lambda(w, w) \mathbb{1}} \star_{z\Lambda} w = w \star_{z\Lambda} e^{tw + \frac{t^2 z}{2} \Lambda(w, w) \mathbb{1}}. \quad (5.5)$$

- iii.) *The star exponential series for $w \in V$ converges absolutely and*

$$\text{Exp}_{\star_{z\Lambda}}(tw) = \sum_{n=0}^{\infty} \frac{t^n}{n!} w \star_{z\Lambda} \dots \star_{z\Lambda} w = e^{tw + \frac{t^2 z}{2} \Lambda(w, w) \mathbb{1}}. \quad (5.6)$$

Proof. We use the continuity of $\star_{z\Lambda}$ and the (absolute) convergence of the exponential series to get

$$\begin{aligned}
\exp(w) \star_{z\Lambda} v &= \sum_{n=0}^{\infty} \frac{1}{n!} w^n \star_{z\Lambda} v \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (w^n v + z\mu \circ P_{\Lambda}(w^n \otimes v) + 0) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (w^n v + nz w^{n-1} \Lambda(w, v)) \\
&= \exp(w)(v + z\Lambda(w, v)),
\end{aligned}$$

since $\mu \circ P_{\Lambda}(\cdot, v)$ is a derivation of the undeformed symmetric tensor product and w is even. The second equation is analogous. For the second part, we first note that $t \mapsto \exp(tw + \frac{t^2 z}{2} \Lambda(w, w) \mathbb{1})$ is real-analytic (entire in the case $\mathbb{K} = \mathbb{C}$) with convergent Taylor expansion around 0 for all $t \in \mathbb{K}$ thanks to Proposition 3.21, *ii.*). We compute the derivative

$$\frac{d}{dt} e^{tw + \frac{t^2 z}{2} \Lambda(w, w) \mathbb{1}} = e^{tw + \frac{t^2 z}{2} \Lambda(w, w) \mathbb{1}} (w + z\Lambda(tw, w)) = e^{tw + \frac{t^2 z}{2} \Lambda(w, w) \mathbb{1}} \star_{z\Lambda} w,$$

using the first part and the fact that $\Lambda(w, w) \mathbb{1}$ is central. Analogously, we can write $w \star_{z\Lambda}$ in front. This shows the second part. Together, this gives

$$\left. \frac{d}{dt} \right|_{t=0} e^{tw + \frac{t^2 z}{2} \Lambda(w, w) \mathbb{1}} = w \star_{z\Lambda} \cdots \star_{z\Lambda} w$$

for the Taylor coefficients of the real-analytic (entire) function $t \mapsto e^{tw + \frac{t^2 z}{2} \Lambda(w, w) \mathbb{1}}$. Since its Taylor series converges absolutely, the last part follows. \square

With this preparation the following statement is now an easy computation:

Proposition 5.6 *Let $R \leq 1$ and let $\varphi \in V'$ be even. If φ is in the image of \sharp then τ_{φ}^* is an inner automorphism of $\hat{\mathcal{W}}_R(V, \star_{z\Lambda})$ for all $z \neq 0$. In fact,*

$$\tau_{\varphi}^*(a) = \text{Exp}_{\star_{z\Lambda}}(w) \star_{z\Lambda} a \star_{z\Lambda} \text{Exp}_{\star_{z\Lambda}}(-w) \quad (5.7)$$

for all $a \in \hat{\mathcal{W}}_R(V, \star_{z\Lambda})$ where $w \in V_0$ is such that $2zw^{\sharp} = \varphi$.

Proof. First we note that the star-exponential function gives a one-parameter group of invertible elements in $\hat{\mathcal{W}}_R(V, \star_{z\Lambda})$ with respect to the star product $\star_{z\Lambda}$. This is clear from the absolute convergence of the star-exponential series. Thus the right hand side of (5.7) defines an inner automorphism of the Weyl algebra. Now consider $v \in V$. Using Lemma 5.5 we compute for $v \in V$

$$\begin{aligned}
\frac{d}{dt} \text{Exp}_{\star_{z\Lambda}}(tw) \star_{z\Lambda} v \star_{z\Lambda} \text{Exp}_{\star_{z\Lambda}}(-tw) &= \text{Exp}_{\star_{z\Lambda}}(tw) \star_{z\Lambda} (w \star_{z\Lambda} v - v \star_{z\Lambda} w) \star_{z\Lambda} \text{Exp}_{\star_{z\Lambda}}(-tw) \\
&= \text{Exp}_{\star_{z\Lambda}}(tw) \star_{z\Lambda} (z\Lambda(w, v) \mathbb{1} - z\Lambda(v, w) \mathbb{1}) \star_{z\Lambda} \text{Exp}_{\star_{z\Lambda}}(-tw) \\
&= 2z\Lambda_-(w, v) \mathbb{1},
\end{aligned}$$

where we use that $\text{Exp}_{\star_{z\Lambda}}(-tw)$ is the $\star_{z\Lambda}$ -inverse of $\text{Exp}_{\star_{z\Lambda}}(tw)$. On the other hand, $t \mapsto \tau_{t\varphi}^*(v) = v + t\varphi(v) \mathbb{1}$ has the derivative

$$\frac{d}{dt} \tau_{t\varphi}^*(v) = \varphi(v) \mathbb{1}.$$

Thus taking w such that $2w\Lambda_-(w, \cdot) = \varphi$, i.e. $2zw^\sharp = \varphi$ shows the claim (5.7) for $a = v$. Now both sides are automorphisms and hence both sides coincide on all $\star_{z\Lambda}$ -polynomials in elements from V . But V together with $\mathbb{1}$ generates $\mathcal{W}_R(V, \star_{z\Lambda})$ according to Corollary 2.10. Thus the two automorphisms coincide on $\mathcal{W}_R(V, \star_{z\Lambda})$. Since both are continuous, they also coincide on the completion $\hat{\mathcal{W}}_R(V, \star_{z\Lambda})$. \square

Remark 5.7 Note that we need the continuity of τ_φ^* in order to show that it coincides with the (obviously continuous) inner automorphism on the right hand side (5.7) on the completion.

5.3 Continuous Equivalences

In the formal setting we have seen that the same antisymmetric part Λ_- yields equivalent deformations, no matter what the symmetric part Λ_+ of Λ is. We extend this now to the analytic framework. The following lemma shows the continuity of the equivalence transformation from Proposition 2.18:

Lemma 5.8 *Let $g: V \times V \rightarrow \mathbb{K}$ be an even symmetric bilinear form. Let $R \geq \frac{1}{2}$ and let p be a seminorm on V with*

$$|g(v, w)| \leq p(v) p(w) \quad (5.8)$$

for all $v, w \in V$. Then we have for all $a \in S^\bullet(V)$

$$p_{R,1}(\Delta_g a) \leq (2p)_{R,1}(a). \quad (5.9)$$

Moreover, for all $\epsilon > 0$ there is a constant $c_{\epsilon,t} > 0$ depending continuously on t with

$$p(e^{t\Delta_g} a) \leq c_{\epsilon,t} p_{R+\epsilon,1}(a). \quad (5.10)$$

Proof. First we extend the operator Δ_g to the whole tensor algebra $T^\bullet(V)$ as usual by setting

$$\tilde{\Delta}_g(v_1 \otimes \cdots \otimes v_n) = \sum_{i < j} (-1)^{v_i(v_1 + \cdots + v_{i-1})} (-1)^{v_j(v_1 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_{j-1})} g(v_i, v_j) v_1 \otimes \cdots \overset{i}{\wedge} \cdots \overset{j}{\wedge} \cdots \otimes v_n$$

on factorizing homogeneous tensor and extending linearly. Then we have

$$S \circ \tilde{\Delta}_g = \Delta_g \circ S \quad (*)$$

as already for P_Λ . With an analogous estimate as for the Poisson bracket we get

$$p^{n-2}(\tilde{\Delta}_g(v_1 \otimes \cdots \otimes v_n)) \leq \sum_{i < j} |g(v_i, v_j)| p(v_1) \cdots \overset{i}{\wedge} \cdots \overset{j}{\wedge} \cdots p(v_n) \leq \frac{n(n-1)}{2} p(v_1) \cdots p(v_n).$$

This implies for all $a_n \in T^n(V)$ the estimate

$$p^{n-2}(\tilde{\Delta}_g a_n) \leq \frac{n(n-1)}{2} p^n(a_n).$$

Thanks to (*) we get the same estimate for $a_n \in S^n(V)$ and Δ_g in place of $\tilde{\Delta}_g$. By induction this results in

$$p^{n-2k}(\Delta_g^k a_n) \leq \frac{n!}{2^k(n-2k)!} p^n(a_n) \quad (**)$$

as long as $n - 2k \geq 0$ and $\Delta_g^k a_n = 0$ for $2k > n$. This gives

$$p_{R,1}(\Delta_g a) = \sum_{n=2}^{\infty} (n-2)!^R p^{n-2}(\Delta_g a_n) \leq \sum_{n=2}^{\infty} (n-2)!^R \frac{n(n-1)}{2} p^n(a_n) \leq \sum_{n=0}^{\infty} n!^R 2^n p^n(a_n)$$

which is the first estimate (5.9). For the second we have to be slightly more efficient with the estimates as a simple iteration of (5.9) would not suffice. We have

$$\begin{aligned}
p_{R,1}(e^{t\Delta_g}a) &\leq \sum_{k=0}^{\infty} \frac{|t|^k}{k!} p_{R,1}(\Delta_g^k a) \\
&\leq \sum_{n,k=0}^{\infty} \frac{|t|^k}{k!} p_{R,1}(\Delta_g^k a_n) \\
&\leq \sum_{n,k,\ell=0}^{\infty} \frac{|t|^k}{k!} \ell!^R p^\ell(\Delta_g^k a_n) \delta_{n-2k,\ell} \\
&\stackrel{(**)}{\leq} \sum_{n,k,\ell=0}^{\infty} \frac{|t|^k}{k!} \ell!^R \frac{n!}{2^k \ell!} p^n(a_n) \delta_{n-2k,\ell} \\
&= \sum_{n,k,\ell=0}^{\infty} \frac{|t|^k}{2^k k!} \ell!^{R-1} n!^{1-R-\epsilon} n!^{R+\epsilon} p^n(a_n) \delta_{n-2k,\ell} \\
&\stackrel{(a)}{\leq} \sum_{n,k,\ell=0}^{\infty} \frac{2^{(1-R-\epsilon)(\ell+2k)} |t|^k}{2^k k!} \ell!^{R-1} \ell!^{1-R-\epsilon} (2k)!^{1-R-\epsilon} n!^{R+\epsilon} p^n(a_n) \delta_{n-2k,\ell} \\
&\stackrel{(b)}{\leq} \sum_{n,k,\ell=0}^{\infty} 2^{(1-R-\epsilon)(\ell+2k)-k} |t|^k \frac{1}{\ell!^\epsilon} 2^{2(1-R-\epsilon)k} k!^{2(1-R-\epsilon)-1} n!^{R+\epsilon} p^n(a_n) \delta_{n-2k,\ell} \\
&\leq \sum_{\ell=0}^{\infty} \frac{2^{(1-R-\epsilon)\ell}}{\ell!^\epsilon} \sum_{k=0}^{\infty} 2^{3k-4Rk-4\epsilon k} |t|^k k!^{1-2R-2\epsilon} \sum_{n=0}^{\infty} n!^{R+\epsilon} p^n(a_n),
\end{aligned}$$

where in (a) we use $n! = (\ell+2k)! \leq 2^{\ell+2k} \ell! (2k)!$ and in (b) we use $(2k)! \leq 2^{2k} k!^2$. Now by assumption $R \geq \frac{1}{2}$ and hence the exponent of $k!$ is strictly negative for $\epsilon > 0$. This shows that the series over k converges and yields a continuous function of $|t|$. The series over ℓ converges as well and gives a constant depending only on ϵ . Finally, the series over n gives the seminorm $p_{R+\epsilon,1}(a)$. \square

We see that the idea of this estimate is rather similar to the one in Lemma 3.10. These estimates provide now the key to establish the equivalences also in the analytical framework:

Proposition 5.9 *Let $R > \frac{1}{2}$ and let $\Lambda, \Lambda': V \times V \rightarrow \mathbb{K}$ be even continuous bilinear forms such that their antisymmetric parts coincide. Then the Weyl algebras $\hat{\mathcal{W}}_R(V, \star_{z\Lambda})$ and $\hat{\mathcal{W}}_R(V, \star_{z\Lambda'})$ are isomorphic via the continuous equivalence transformation*

$$e^{z\Delta_g}(a \star_{z\Lambda} b) = (e^{z\Delta_g}a) \star_{z\Lambda'} (e^{z\Delta_g}b), \quad (5.11)$$

where $g = \Lambda' - \Lambda = \Lambda'_+ - \Lambda_+$ and $a, b \in \hat{\mathcal{W}}_R(V)$.

Proof. First we note that the continuity of Λ and Λ' implies the continuity of g . Moreover, to test the continuity of the map $e^{z\Delta_g}$ it clearly suffices to consider only those seminorms $p_{R-\epsilon,1}$ of $\hat{\mathcal{W}}_R(V)$ with p being a seminorm such that (5.8) holds. Thus we can apply Lemma 5.8 to conclude that $e^{z\Delta_g}$ is continuous on $\mathcal{W}_R(V)$ and hence extends to a continuous endomorphism of $\hat{\mathcal{W}}_R(V)$ as well. Then the relation (5.11) holds for all $a, b \in \mathcal{W}_R(V)$ by Proposition 2.18 and hence for all $a, b \in \hat{\mathcal{W}}_R(V)$ by

continuity. Finally, for a fixed $a \in \hat{\mathcal{W}}_R(V)$ the exponential series

$$e^{t\Delta_g}a = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta_g^k a$$

converges absolutely in the topology of $\hat{\mathcal{W}}_R(V)$. Indeed, this follows from the estimate in the proof of Lemma 5.8. Thus for $z, w \in \mathbb{K}$ we get $e^{z\Delta_g} \circ e^{w\Delta_g} = e^{(z+w)\Delta_g}$ at once. This shows that $e^{z\Delta_g}$ is indeed invertible and hence a continuous isomorphism with continuous inverse $e^{-z\Delta_g}$. \square

Remark 5.10 Note that $e^{z\Delta_g}$ maps $\mathcal{W}_R(V)$ into itself. Hence also the (non-completed) Weyl algebras $\mathcal{W}_R(V, \star_{z\Lambda})$ and $\mathcal{W}_R(V, \star_{z\Lambda'})$ are isomorphic via $e^{z\Delta_g}$.

Remark 5.11 Putting together the results from Proposition 5.2 and Proposition 5.9 we see that the isomorphism class of the Weyl algebra $\mathcal{W}_R(V, \star_{z\Lambda})$ is determined by the isomorphism class of the *antisymmetric* part Λ_- alone.

Remark 5.12 In the situation of Proposition 3.26 we see that also in the \ast -algebra case, two Weyl algebras $\mathcal{W}_R(V, \star_{i\frac{\hbar}{2}\Lambda})$ and $\mathcal{W}_R(V, \star_{i\frac{\hbar}{2}\Lambda'})$ are \ast -equivalent via an equivalence transformation which is now a \ast -isomorphism provided that the antisymmetric parts of Λ and Λ' coincide. The reason is that the symmetric parts are necessarily purely imaginary.

5.4 Finite Dimensional V

In the finite-dimensional case the situation is very simple: first we note that there is only one Hausdorff locally convex topology on V and all bilinear maps are continuous. In this situation we get a defining system of continuous seminorms for the topology of $\mathcal{W}_R(V)$ very easily:

Lemma 5.13 *Let V be finite-dimensional and let p be a norm on V . Then the norms $\{p_{R-\epsilon,1}\}_{\epsilon>0}$ yield a defining system of seminorms for $\mathcal{W}_R(V)$.*

Proof. Let q be an arbitrary seminorm on V . Then there is a constant $c > 0$ with $q \leq cp$ since p is a norm and we are in finite dimensions. Then we have $q^n \leq c^n p^n$. Now fix $\epsilon' > 0$ with $\epsilon' < \epsilon$ and let $C > 0$ be a constant such that $c^n \leq Cn!^{\epsilon-\epsilon'}$ for all $n \in \mathbb{N}$. Then we have

$$q_{R-\epsilon,1}(a) = \sum_{n=0}^{\infty} n!^{R-\epsilon} q^n(a_n) \leq \sum_{n=0}^{\infty} n!^{R-\epsilon} c^n p^n(a_n) \leq C \sum_{n=0}^{\infty} n!^{R-\epsilon'} p^n(a_n) = C p_{R-\epsilon',1}(a).$$

This shows that we can estimate every seminorm of the form $q_{R-\epsilon,1}$ by a suitable $p_{R-\epsilon',1}$. \square

Let $V = V_0 \oplus V_1$ be finite-dimensional and real. Moreover, let $\Lambda: V \times V \rightarrow \mathbb{R}$ be antisymmetric and even. Then $\Lambda = \Lambda_0 + \Lambda_1$ with

$$\Lambda_0: V_0 \times V_0 \rightarrow \mathbb{R} \quad \text{and} \quad \Lambda_1: V_1 \times V_1 \rightarrow \mathbb{R}, \quad (5.12)$$

such that Λ_0 is an antisymmetric bilinear form on V_0 and Λ_1 is a symmetric bilinear form on V_1 . By the linear Darboux Theorem we can find a basis $q_1, \dots, q_d, p_1, \dots, p_d, c_1, \dots, c_k$ of V_0 such that the only nontrivial pairing is

$$\Lambda_0(q_i, p_j) = \delta_{ij} = -\Lambda_0(p_j, q_i). \quad (5.13)$$

For the odd part, we can find a basis $e_1, \dots, e_r, f_1, \dots, f_s, x_1, \dots, x_t$ with the only nontrivial pairings

$$\Lambda_1(e_i, e_j) = \delta_{ij} \quad \text{and} \quad \Lambda_1(f_i, f_j) = -\delta_{ij}. \quad (5.14)$$

Here $\dim V_0 = 2d + k$ and $\dim V_1 = r + s + t$. Then Λ_0 is symplectic iff $k = 0$ and Λ_1 is an (indefinite) inner product iff $t = 0$, its signature is then given by (r, s) . If we use Λ directly for building the star product $\star_{z\Lambda}$ in this case then we obtain the usual Weyl-Moyal star product for the even part and a Clifford multiplication for the odd part. Thus the numbers d, k, r, s, t encode the isomorphism class of $\mathcal{W}_R(V, \star_{z\Lambda})$ in the finite-dimensional case. The complex case is analogous.

Let us now use this simple classification to compare our general construction with a previous construction in finite dimensions: the construction of a convergent Wick star product in [3, 4]. Here we are in the real symplectic situation with $V_{\mathbb{R}} = \mathbb{R}^{2d}$ and its canonical symplectic form. We use compatible complex coordinates and write the complexified symmetric algebra $S^\bullet(V)$ as polynomial algebra in the variables $z^1, \dots, z^d, \bar{z}^1, \dots, \bar{z}^d$, i.e.

$$S^\bullet(V) = \mathbb{C}[z^1, \dots, z^d, \bar{z}^1, \dots, \bar{z}^d], \quad (5.15)$$

where the reality structure inherited from $V = V_{\mathbb{R}} \otimes \mathbb{C}$ simply means that $\overline{z^k} = \bar{z}^k$ for all $k = 1, \dots, d$. The Wick star product

$$f \star_{\text{Wick}} g = \sum_{N=0}^{\infty} \frac{(2\hbar)^{|N|}}{N!} \frac{\partial^{|N|} f}{\partial z^N} \frac{\partial^{|N|} g}{\partial \bar{z}^N} \quad (5.16)$$

from [3, 4] can then be written as $\star_{\text{Wick}} = \star_{\frac{i\hbar}{2}\Lambda}$ with Λ given by

$$\Lambda(z^k, \bar{z}^\ell) = 4\delta_{k\ell}, \quad (5.17)$$

and all other pairings trivial. In [4, Thm. 3.19] it was shown that the previously constructed locally convex topology for the Wick star product from [3] can be described as follows: write $a \in S^\bullet(V)$ as Taylor polynomial

$$a = \sum_{I, J=0}^{\infty} a_{IJ} \frac{z^I \bar{z}^J}{I! J!}. \quad (5.18)$$

Then the defining system of seminorms is given by

$$\|a\|_\epsilon = \sup_{I, J} \frac{|a_{IJ}|}{|I + J|!^\epsilon}, \quad (5.19)$$

where $\epsilon > 0$. With other words, the Taylor coefficients a_{IJ} have *sub-factorial* growth with respect to the multiindices I and J . Note that in [4] an addition factor $(2\hbar)^{|I|+|J|}$ is present in the denominator in (5.18). But clearly such an exponential contribution will not change the sub-factorial growth properties at all. Therefore the seminorms (5.19) give the same topology as the one in [4].

Proposition 5.14 *The locally convex topology on $S^\bullet(V)$ induced by the seminorms $\{\|\cdot\|_\epsilon\}_{\epsilon>0}$ coincides with the topology of the Weyl algebra $\mathcal{W}_R(V)$ for $R = 1$.*

Proof. To get the combinatorics of the Taylor and tensor coefficients right, we note that $a_n \in S^n(V) \subseteq T^n(V)$ can be written as

$$a_n = \sum_{\alpha_1, \dots, \alpha_n} a_{\alpha_1 \dots \alpha_n} e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n},$$

where the basis vectors of $V = \mathbb{R}^{2d} \otimes \mathbb{C}$ are

$$e_1 = z^1, \dots, e_d = z^d, \quad \text{and} \quad e_{\bar{1}} = \bar{z}^1, \dots, e_{\bar{d}} = \bar{z}^d$$

and the summation indices run from 1 to d and $\bar{1}$ to \bar{d} . From $a_n = \mathcal{S} a_n$ we see that the coefficients $a_{\alpha_1 \dots \alpha_n}$ are totally symmetric. Comparing with (5.18) we see that

$$\frac{a_{IJ}}{I! J!} = \sum_{\alpha \in (I, J)} a_{\alpha_1 \dots \alpha_n},$$

where the summation runs over all those n -tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ containing i_1 times the index 1, \dots , i_d times the index d , j_1 times the index $\bar{1}$, \dots , and j_d times the index \bar{d} . Since the coefficients are totally symmetric we get

$$\frac{|a_{IJ}|}{I!J!} = \sum_{\alpha \in (I,J)} |a_{\alpha_1 \dots \alpha_n}|. \quad (*)$$

The first estimate we need is now

$$\begin{aligned} \|a\|_\epsilon &= \sup_{I,J} \frac{|a_{IJ}|}{|I+J|^\epsilon} \\ &= \sup_n \frac{1}{n!^\epsilon} \sup_{\substack{I,J \\ |I+J|=n}} \sum_{\alpha \in (I,J)} |a_{\alpha_1 \dots \alpha_n}| I!J! \\ &\leq \sum_{n=0}^{\infty} \frac{1}{n!^\epsilon} \sum_{\alpha_1, \dots, \alpha_n} |a_{\alpha_1 \dots \alpha_n}| n! \\ &= p_{1-\epsilon,1}(a), \end{aligned}$$

where p is the norm on $\mathbb{R}^{2d} \otimes \mathbb{C}$ given by

$$p(a) = \sum_{\alpha} |a_{\alpha}|,$$

i.e. the ℓ^1 -norm with respect to the basis $e_1, \dots, e_{\bar{d}}$. The reason to chose this ℓ^1 -norm is that it behaves most nicely for the tensor product. For the other direction we take the same norm p and chose again a $0 < \epsilon' < \epsilon$. Then

$$\begin{aligned} p_{1-\epsilon,1}(a) &= \sum_{n=0}^{\infty} n^{1-\epsilon} \sum_{\alpha_1, \dots, \alpha_n} |a_{\alpha_1 \dots \alpha_n}| \\ &= \sum_{n=0}^{\infty} n^{1-\epsilon} \sum_{|I+J|=n} \frac{|a_{IJ}|}{I!J!} \\ &\leq \sum_{n=0}^{\infty} n^{1-\epsilon} \sum_{|I+J|=n} \frac{\|a\|_{\epsilon'} n!^{\epsilon'}}{I!J!} \\ &= \|a\|_{\epsilon'} \sum_{n=0}^{\infty} \frac{(2d)^n}{n!^{\epsilon-\epsilon'}}, \end{aligned}$$

which gives an estimate $p_{R-\epsilon,1}(a) \leq c\|a\|_{\epsilon'}$ with c being the above convergent series. Since by Lemma 5.13 it suffices to consider one norm p for V , this finishes the proof. \square

Remark 5.15 The construction in [3, 4] still had the flavour of being rather ad-hoc. Thanks to Proposition 5.14 and the more conceptual definition of the topology of the Weyl algebra $\mathcal{W}_R(V)$ in Definition 3.13 this flaw disappears. Also, the choice of the Wick star product is not essential as on $V_{\mathbb{R}} = \mathbb{R}^{2d}$ there is only one symplectic form, see Remark 5.11.

6 An Example: the Peierls Bracket and Free QFT

In this last section we discuss a first example in infinite dimensions: the Poisson bracket and the corresponding Weyl algebra underlying a free, i.e. linear, field theory. Our main focus here is the precise definition of the relevant locally convex topologies as well as the global aspects of the construction. One has essentially two possibilities for the Poisson structure: the canonical Poisson structure built on a Hamiltonian formulation using the initial value problem and the covariant Poisson structure, also called the Peierls bracket, built on a Lagrangian approach. In the following, we will exclusively work in the smooth category, all manifolds and bundles will be \mathcal{C}^∞ .

The material on the Cauchy problem on globally hyperbolic spacetimes is standard and can be found e.g. in the book [1]. The comparison of the canonical and the covariant Poisson brackets is folklore and was taken from [25, Sect. 4.4]. For a much more far-reaching discussion of the Peierls bracket including also non-linear field equations we refer to [7]: in fact, it would be a very interesting project to combine the results from [7] on the classical side with the nuclear Weyl algebra quantization to obtain the corresponding quantum side.

6.1 The Geometric Framework

We consider an n -dimensional connected Lorentz manifold (M, g) with a Lorentz metric g of signature $(+, -, \dots, -)$. The important concept we need is the causal structure: first we require that (M, g) is time-orientable and time-oriented. Then one defines the causal future $J_M^+(p)$ of a point $p \in M$ to be the set of those points which can be reached by a future-directed causal curve. Analogously, $J_M^-(p)$ denotes the causal past of p . For two points $p, q \in M$ one defines the diamond $J_M(p, q) = J_M^+(p) \cap J_M^-(q)$. For an arbitrary subset $A \subseteq M$ we set

$$J_M^\pm(A) = \bigcup_{p \in A} J_M^\pm(p) \quad \text{and} \quad J_M(A) = J_M^+(A) \cup J_M^-(A). \quad (6.1)$$

A time-oriented Lorentz manifold is called *globally hyperbolic* if it is causal, i.e. there are no closed causal loops, and if all diamonds are compact. A first consequence is that $J_M^\pm(p)$ is always a closed subset of M . In the following we always assume that (M, g) is globally hyperbolic. A celebrated theorem of Bernal and Sánchez, refining a topological statement of Geroch, states that this is equivalent to the existence of a smooth spacelike Cauchy surface

$$\iota: \Sigma \longrightarrow M \quad (6.2)$$

together with a smooth Cauchy temporal function $t \in \mathcal{C}^\infty(M)$, i.e. the gradient of t is future directed and time-like everywhere and the level sets of t are smooth spacelike Cauchy surfaces for all times. Moreover, M is diffeomorphic to the product manifold $\mathbb{R} \times \Sigma$ with the metric being

$$g = \beta dt^2 - g_t, \quad (6.3)$$

where $\beta \in \mathcal{C}^\infty(\mathbb{R} \times \Sigma)$ is positive and g_t is a Riemannian metric on Σ smoothly depending on t . Finally, the Cauchy temporal function t can be chosen in such a way that Σ is the $t = 0$ level set. For a detailed discussion see the review [21].

Since M is diffeomorphic to $\mathbb{R} \times \Sigma$ we get a global vector field $\frac{\partial}{\partial t}$ on M . Normalizing this to a unit vector field gives

$$\mathbf{n} = \frac{1}{\sqrt{\beta}} \frac{\partial}{\partial t} \in \Gamma^\infty(TM), \quad (6.4)$$

which is a future-directed time-like unit vector field such that it is normal to every level surface of the Cauchy temporal function. In particular $\iota^\# \mathbf{n} \in \Gamma^\infty(TM_\Sigma)$ will be normal to the Cauchy surface

Σ . Here $TM_\Sigma = \iota^\# TM$ is the restriction (pull-back via ι) of the tangent bundle to Σ and $\iota^\# \mathbf{n} = \mathbf{n}|_\Sigma$ is the pull-back of \mathbf{n} to Σ .

The fields we are interested in will be modeled by sections of a vector bundle over M : we require the vector bundle $E \rightarrow M$ to be real and equipped with a fiber metric h , not necessarily positive definite but non-degenerate. The dynamics of the field is now governed by a second order differential operator

$$D \in \text{DiffOp}^2(E) \quad (6.5)$$

with the following property: there is a metric connection ∇^E for (E, h) and a zeroth order differential operator $B \in \text{DiffOp}^0(E) = \Gamma^\infty(\text{End}(E))$ such that

$$D = \square^\nabla + B, \quad (6.6)$$

where \square^∇ denotes the d'Alembert operator obtained from ∇^E and pairing with the metric g . In particular, D is *normally hyperbolic*. Conversely, note that for a normally hyperbolic differential operator there is a unique connection ∇^E and a unique $B \in \Gamma^\infty(\text{End}(E))$ such that (6.6) holds, see e.g. [1, Lem. 1.5.5]. Thus the only additional requirement we need is that ∇^E is also metric with respect to h .

Let $\mu_g \in \Gamma^\infty(|\Lambda^n T^*M|)$ be the canonical metric density induced by g which we shall use for various integrations over M . First we can use μ_g to define the *transpose* of a differential operator $D \in \text{DiffOp}^\bullet(E)$ to be the unique differential operator $D^T \in \text{DiffOp}^\bullet(E^*)$ such that

$$\int_M (D^T \varphi) \cdot u \mu_g = \int_M \varphi \cdot (Du) \mu_g \quad (6.7)$$

for all $\varphi \in \Gamma^\infty(E^*)$ and $u \in \Gamma^\infty(E)$, at least one of them having compact support. Here \cdot means pointwise natural pairing. Note that D^T depends on the choice of μ_g and has the same order as D . Taking into account also the fiber metric h we can define the *adjoint* of D to be the unique differential operator $D^* \in \text{DiffOp}^\bullet(E)$, again of the same order as D , such that

$$\int_M h(D^* u, v) \mu_g = \int_M h(u, Dv) \mu_g \quad (6.8)$$

for $u, v \in \Gamma^\infty(E)$, at least one of them having compact support. Denoting the musical isomorphisms induced by h by $\sharp: E^* \rightarrow E$ and $\flat: E \rightarrow E^*$ as usual, we get $D^* u = (D^T u^\flat)^\sharp$ for $u \in \Gamma^\infty(E)$.

This allows now to formulate the last requirement on $D = \square^\nabla + B$, namely we need D to be a *symmetric* operator, i.e.

$$D^* = D. \quad (6.9)$$

Since the connection ∇^E is required to be metric, it is easy to see that (6.9) is equivalent to $B^* = B$.

6.2 The Wave Equation and Green Operators

Let D be a normally hyperbolic differential operator as before. Then the *wave equation* we are interested in is simply given by

$$Du = 0 \quad (6.10)$$

for a section u of E . Depending on the regularity of u we can interpret (6.10) as a pointwise equation or as an equation in a distributional sense. Dualizing, we have the corresponding wave equation

$$D^T \varphi = 0 \quad (6.11)$$

for a section φ of the dual bundle E^* .

Under our general assumption that (M, g) is globally hyperbolic we have the existence and uniqueness of advanced and retarded *Green operators*

$$G_M^\pm : \Gamma_0^\infty(E) \longrightarrow \Gamma^\infty(E) \quad (6.12)$$

for D . This means that there are unique, linear, and continuous maps G_M^\pm such that

$$DG_M^\pm = \text{id}_{\Gamma_0^\infty(E)} = G_M^\pm D|_{\Gamma_0^\infty(E)} \quad (6.13)$$

and

$$\text{supp } G_M^\pm \subseteq J_M^\pm(\text{supp } u) \quad (6.14)$$

for all $u \in \Gamma_0^\infty(E)$. The continuity refers to the usual LF and Fréchet topologies of $\Gamma_0^\infty(E)$ and $\Gamma^\infty(E)$, respectively. Using the volume density μ_g we can identify the distributional sections $\Gamma^{-\infty}(E^*)$ with the dual $\Gamma_0^\infty(E)'$ and $\Gamma_0^{-\infty}(E^*)$ becomes identified with the dual $\Gamma^\infty(E)'$.

We need the following space of sections: Let $K \subseteq M$ be compact. Then denote by $\Gamma_{J_M(K)}^\infty(E)$ those sections in $\Gamma^\infty(E)$ with $\text{supp } u \subseteq J_M(K)$. Since on a globally hyperbolic spacetime $J_M(K)$ is a closed subset, $\Gamma_{J_M(K)}^\infty(E) \subseteq \Gamma^\infty(E)$ is a closed subspace and hence a Fréchet space itself. Moreover, for $K \subseteq K'$ we have $\Gamma_{J_M(K)}^\infty(E) \subseteq \Gamma_{J_M(K')}^\infty(E)$ and the thereby induced topology on $\Gamma_{J_M(K)}^\infty(E)$ coincides with the original. Hence we can consider the inductive limit

$$\Gamma_{\text{sc}}^\infty(E) = \bigcup_{\substack{K \subseteq M \\ K \text{ compact}}} \Gamma_{J_M(K)}^\infty(E) \quad (6.15)$$

of those smooth sections of E which have compact support in spacelike directions. It is a strict inductive limit, and since we can exhaust M with a sequence of compact subsets, it is a countable strict inductive limit, endowing $\Gamma_{\text{sc}}^\infty(E)$ with a LF topology. The continuity statement (6.12) can then be sharpened to the statement that

$$G_M^\pm : \Gamma_0^\infty(E) \longrightarrow \Gamma_{\text{sc}}^\infty(E) \quad (6.16)$$

is continuous. In fact, this follows in a straightforward manner from the continuity of (6.12) and the causality condition (6.14).

Remark 6.1 Being a close subspace of a nuclear Fréchet space, $\Gamma_{J_M(K)}^\infty(E)$ is nuclear itself. Hence the countable strict inductive limit $\Gamma_{\text{sc}}^\infty(E)$ is again nuclear by [20, Cor. 21.2.3].

We consider now the *propagator* which is defined by

$$G_M = G_M^+ - G_M^- : \Gamma_0^\infty(E) \longrightarrow \Gamma_{\text{sc}}^\infty(E), \quad (6.17)$$

for which one has the following crucial properties: the sequence

$$0 \longrightarrow \Gamma_0^\infty(E) \xrightarrow{D} \Gamma_0^\infty(E) \xrightarrow{G_M} \Gamma_{\text{sc}}^\infty(E) \xrightarrow{D} \Gamma_{\text{sc}}^\infty(E) \quad (6.18)$$

of continuous linear maps is *exact*. Note that the exactness relies crucially on the assumption that (M, g) is globally hyperbolic.

The Green operators can now be used to give a solution to the Cauchy problem of the wave equation (6.10). For the time $t = 0$ level surface Σ we want to specify initial conditions $u_0, \dot{u}_0 \in \Gamma_0^\infty(E_\Sigma)$. Then we want to find a section $u \in \Gamma^\infty(E)$ with

$$Du = 0, \quad \iota^\# u = u_0, \quad \text{and} \quad \iota^\# \nabla_n^E u = \dot{u}_0. \quad (6.19)$$

A core result in the globally hyperbolic case is that this is indeed a well-posed Cauchy problem: for any (u_0, \dot{u}_0) we have a unique solution u of the Cauchy problem (6.19) such that the map

$$\Gamma_0^\infty(E_\Sigma) \oplus \Gamma_0^\infty(E_\Sigma) \ni (u_0, \dot{u}_0) \mapsto u \in \Gamma_{sc}^\infty(E) \quad (6.20)$$

is continuous and $\text{supp } u \subseteq J_M(\text{supp } u_0 \cup \text{supp } \dot{u}_0)$. Moreover, this solution u can be characterized by the formula

$$\int_M \varphi \cdot u \, \mu_g = \int_\Sigma \left(\iota^\#(\nabla_n^E F_M(\varphi)) \cdot u_0 - \iota^\#(F_M(\varphi)) \cdot \dot{u}_0 \right) \mu_\Sigma, \quad (6.21)$$

where F_M is the propagator of D^T and $\varphi \in \Gamma_0^\infty(E^*)$. The density μ_Σ is the one induced by μ_g .

Since we assume that $D^* = D$ we have a last property of the Green operators, namely

$$(G_M^\pm)^* = G_M^\mp \quad \text{and} \quad G_M^* = -G_M. \quad (6.22)$$

The antisymmetry of the propagator will play a crucial role in the definition of the covariant Poisson bracket. Moreover, in this case the relation between G_M^\pm and F_M^\pm is

$$G_M^\pm(\varphi^\sharp) = (F_M^\pm(\varphi))^\sharp \quad (6.23)$$

for all $\varphi \in \Gamma_0^\infty(E^*)$.

The results and their proofs of this section as well as many more additional features of the Cauchy problem of the wave equation on a globally hyperbolic spacetime can be found in the beautiful book [1], some additional remarks can be found in the lecture notes [25].

6.3 The Canonical Poisson Algebra

The canonical i.e. Hamiltonian approach uses an algebra of functions on the initial data, which constitute the classical phase space

$$\mathcal{P}_\Sigma = \Gamma_0^\infty(E_\Sigma) \oplus \Gamma_0^\infty(E_\Sigma). \quad (6.24)$$

We view \mathcal{P}_Σ as symplectic vector space via the symplectic form

$$\omega_\Sigma((u_0, \dot{u}_0), (v_0, \dot{v}_0)) = \int_\Sigma (h_\Sigma(u_0, \dot{v}_0) - h_\Sigma(\dot{u}_0, v_0)) \, \mu_\Sigma, \quad (6.25)$$

where h_Σ is the restriction of h to $E|_\Sigma$. We have the following basic result:

Lemma 6.2 *The two-form ω_Σ on \mathcal{P}_Σ is antisymmetric, non-degenerate, and continuous.*

Proof. The non-degeneracy and the antisymmetry are clear. For the continuity we can rely on several standard arguments: first we note that every vector bundle can be written as a subbundle of a suitable trivial vector bundle $\Sigma \times \mathbb{R}^N$. This gives an identification $\Gamma_0^\infty(E_\Sigma) \subseteq \Gamma_0^\infty(\Sigma \times \mathbb{R}^N)$ as a *closed embedded* subspace. We can extend h_Σ in some way to a smooth fiber metric on the trivial bundle and this way, ω_Σ is just the restriction of the corresponding symplectic form on $\Gamma^\infty(\Sigma \times \mathbb{R}^N)$. Thus it suffices to consider a trivial bundle from the beginning. There we have $\Gamma_0^\infty(\Sigma \times \mathbb{R}^N) \cong \mathcal{C}_0^\infty(\Sigma)^N$. Thus we have to show the continuity of a bilinear map of the form

$$\mathcal{C}_0^\infty(\Sigma)^N \times \mathcal{C}_0^\infty(\Sigma)^N \ni ((u_i), (v_i)) \mapsto \sum_{i,j=1}^N u_i H_{ij} v_j \in \mathcal{C}_0^\infty(\Sigma), \quad (*)$$

where $H_{ij} \in \mathcal{C}^\infty(\Sigma)$. But since the multiplication of compactly supported functions is continuous (and not just separately continuous) the continuity of $(*)$ follows. The final integration needed for (6.25) is continuous as well. \square

Note that it is obvious that ω_Σ is separately continuous, however, we are interested in continuity. In the case where Σ is compact, this would follow directly from separate continuity as then $\Gamma_0^\infty(E_\Sigma) = \Gamma^\infty(E_\Sigma)$ is a Fréchet space.

The idea is now to look at certain polynomial functions on \mathcal{P}_Σ and endow them with the Poisson bracket originating from ω_Σ . It turns out that the symmetric algebra over the dual \mathcal{P}'_Σ will be too big and problematic when it comes to the comparison with the covariant Poisson structure. Hence we decide here for a rather small piece of all polynomials, namely for the symmetric algebra over

$$V_\Sigma = \Gamma_0^\infty(E_\Sigma^*) \oplus \Gamma_0^\infty(E_\Sigma^*). \quad (6.26)$$

Using the density μ_Σ we can indeed pair elements from V_Σ with points in \mathcal{P}_Σ :

Lemma 6.3 *The integration*

$$(\varphi_0, \dot{\varphi}_0)(u_0, \dot{u}_0) = \int_\Sigma (\varphi_0 \cdot u_0 + \dot{\varphi}_0 \cdot \dot{u}_0) \mu_\Sigma \quad (6.27)$$

provides a continuous bilinear pairing between V_Σ and \mathcal{P}_Σ .

The proof of the continuity is analogous to the one in Lemma 6.2. In particular, we can view points in \mathcal{P}_Σ as elements of the dual of V_Σ and vice versa.

Lemma 6.4 *The symplectic form ω_Σ induces a non-degenerate antisymmetric continuous bilinear form*

$$\Lambda_\Sigma: V_\Sigma \times V_\Sigma \longrightarrow \mathbb{R}, \quad (6.28)$$

explicitly given by

$$\Lambda_\Sigma((\varphi_0, \dot{\varphi}_0), (\psi_0, \dot{\psi}_0)) = \int_\Sigma \left(h_\Sigma^{-1}(\varphi_0, \dot{\psi}_0) - h_\Sigma^{-1}(\dot{\varphi}_0, \psi_0) \right) \mu_\Sigma. \quad (6.29)$$

Here h_Σ^{-1} stands for the induced fiber metric on E_Σ^* and the Poisson bracket is determined by ω_Σ in the sense that the Hamiltonian vector field of the linear function $(\varphi_0, \dot{\varphi}_0)$ on \mathcal{P}_Σ is determined via ω_Σ and the Poisson bracket is determined by the Hamiltonian vector field as usual.

We can now use the Poisson bracket $\{\cdot, \cdot\}_{\Lambda_\Sigma}$ for the symmetric algebra $\mathbf{S}^\bullet(V_\Sigma)$ as described in Section 2.2 together with its quantization given by the star product $\star_\Sigma = \star_{\frac{i\hbar}{2}\Lambda_\Sigma}$ for the corresponding nuclear Weyl algebra $\mathcal{W}_R(V_\Sigma \otimes \mathbb{C})$. This will be the canonically quantized model of our Hamiltonian picture of the field theory. Since we started with a real vector bundle, the resulting Weyl algebra carries the complex conjugation as $*$ -involution. Note however that up to now we only described the kinematic part, the field equation did not yet enter at all.

6.4 The Covariant Poisson Algebra

As the covariant “phase space” we take simply all possible field configurations on the spacetime, i.e.

$$\mathcal{P}_{\text{cov}} = \Gamma_{\text{sc}}^\infty(E), \quad (6.30)$$

whether or not they satisfy the wave equation. This will not be a symplectic vector space in any reasonable sense as \mathcal{P}_{cov} contains all the unwanted field configurations as well. Nevertheless, and this is perhaps the surprising observation, the symmetric algebra over its dual allows for a Poisson bracket: again, we take only a small part of the dual, namely $\Gamma_0^\infty(E^*)$, where we evaluate $\varphi \in \Gamma_0^\infty(E^*)$ on $u \in \Gamma_{\text{sc}}^\infty(E)$ by means of the integration with respect to μ_g as usual. As before, we denote this integration simply by $\varphi(u)$.

The Poisson bracket will then be determined by a bilinear form on $\Gamma_0^\infty(E^*)$ as before. Using the propagator F_M of D^T we define

$$\Lambda_{\text{cov}}(\varphi, \psi) = \int_M h^{-1}(F_M(\varphi), \psi) \mu_g. \quad (6.31)$$

Note that the compact support of ψ makes this integration well-defined. Moreover, we have the following property:

Lemma 6.5 *The bilinear form $\Lambda_{\text{cov}}: \Gamma_0^\infty(E^*) \times \Gamma_0^\infty(E^*) \longrightarrow \mathbb{R}$ is antisymmetric and continuous.*

Proof. The antisymmetry is clear since D^T is symmetric and hence (6.22) applies also to F_M . The continuity is slightly more involved: first we note that $F_M: \Gamma_0^\infty(E^*) \longrightarrow \Gamma^\infty(E^*)$ is continuous by the continuity of the Green operators F_M^\pm . Next, we use the fact that the inclusion $\Gamma^\infty(E^*) \longrightarrow \Gamma_0^\infty(E^*)'$ given by the integration with respect to μ_g using h^{-1} is also continuous where we equip the dual $\Gamma_0^\infty(E^*)'$ with the *strong* topology. This shows that the corresponding “musical” homomorphism

$$\sharp_{\text{cov}}: \Gamma_0^\infty(E^*) \ni \varphi \mapsto \Lambda_{\text{cov}}(\varphi, \cdot) \in \Gamma_0^\infty(E^*)'$$

is continuous with respect to the LF and the strong topology, respectively. Hence the Kernel Theorem for the nuclear space $\Gamma_0^\infty(E^*)$ states that $\Lambda_{\text{cov}}(\cdot, \cdot)$ is a distribution on the Cartesian product, or, equivalently, a continuous bilinear map, see e.g. [20, Thm. 21.6.9]. \square

Remark 6.6 Contrary to the continuity of Λ_Σ the above argument does not simplify for a compact Cauchy surface Σ since a globally hyperbolic spacetime is always non-compact.

Definition 6.7 (Covariant Poisson algebra) *The covariant Poisson algebra for D is the symmetric algebra $S^\bullet(V_{\text{cov}})$, where $V_{\text{cov}} = \Gamma_0^\infty(E^*)$, with the constant Poisson bracket $\{\cdot, \cdot\}_{\text{cov}}$ coming from Λ_{cov} .*

This is indeed a Poisson algebra with a continuous Poisson bracket if we endow it with one of the topologies discussed in Section 3.1, see Proposition 3.9 and Remark 3.15. A first and heuristic appearance of this Poisson bracket in a very particular case seems to be [22].

From our general theory we know that the corresponding *covariant Weyl algebra* $\mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C}, \star_{\text{cov}})$ with the *covariant star product* $\star_{\text{cov}} = \star_{\frac{i\hbar}{2}\Lambda_{\text{cov}}}$ is a nuclear \ast -algebra with respect to the complex conjugation, where as usual $R > \frac{1}{2}$.

As a first result we note that Λ_{cov} is now degenerate. In fact, we can determine its degeneracy space explicitly [25, Lem. 4.4.18]:

Lemma 6.8 *Let $\varphi \in \Gamma_0^\infty(E^*)$. Then the following statements are equivalent:*

- i.) φ is a Casimir element of $S^\bullet(V_{\text{cov}})$, i.e. $\{\varphi, \cdot\}_{\text{cov}} = 0$.
- ii.) φ vanishes on solutions $u \in \Gamma_{\text{sc}}^\infty(E)$, i.e. we have

$$\int_M \varphi \cdot u \mu_g = 0 \quad \text{whenever} \quad Du = 0. \quad (6.32)$$

- iii.) $\varphi \in \ker F_M$.

Proof. Assume $\{\varphi, \cdot\}_{\text{cov}} = 0$ then for all $\psi \in \Gamma_0^\infty(E^*)$ we have $0 = \{\varphi, \psi\}_{\text{cov}} = \int_M h^{-1}(F_M(\varphi), \psi) \mu_g$. Since h^{-1} is non-degenerate this implies $F_M(\varphi) = 0$. Next, assume $F_M(\varphi) = 0$. Then we know $\varphi = D^T \chi$ for some $\chi \in \Gamma_0^\infty(E^*)$ by (6.18) applied to D^T . Thus (6.32) follows by definition of D^T as in (6.7). Finally, assume (6.32) and let $\psi \in \Gamma_0^\infty(E^*)$ be arbitrary. Then $(F_M(\psi))^\sharp = G_M(\psi^\sharp)$ by (6.23) and it solves the wave equation $DG_M(\psi^\sharp) = 0$ by (6.18). Thus $\{\varphi, \psi\}_{\text{cov}} = 0$ follows. Since $S^\bullet(V_{\text{cov}})$ is generated by V_{cov} and $\{\varphi, \cdot\}_{\text{cov}}$ is a derivation, $\{\varphi, \cdot\}_{\text{cov}} = 0$ follows. \square

Since the elements of $\ker F_M \subseteq \Gamma_0^\infty(E^*)$ are Casimir elements, the ideal they generate inside $S^\bullet(V_{\text{cov}})$ is a Poisson ideal. It turns out that it is even a two-sided ideal with respect to \star_{cov} :

Lemma 6.9 *Let $\langle \ker F_M \rangle \subseteq \mathbf{S}^\bullet(V_{\text{cov}})$ be the ideal generated by $\ker F_M$ with respect to the symmetric tensor product. Then we have:*

- i.) $\langle \ker F_M \rangle$ is a Poisson ideal with respect to $\{\cdot, \cdot\}_{\text{cov}}$.*
- ii.) $\langle \ker F_M \rangle \otimes \mathbb{C} \subseteq \mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C})$ is a \star -ideal for \star_{cov} , in fact generated by $\ker F_M$.*

Proof. The first part is clear by Lemma 6.8, *i.*). Now let $\Phi \in \mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C})$ be an arbitrary tensor and let $\varphi \in \Gamma_0^\infty(E^*)$. Then we have

$$\Phi \star_{\text{cov}} \varphi = \Phi \varphi + \frac{i\hbar}{2} \{\Phi, \varphi\}_{\text{cov}} \quad \text{and} \quad \varphi \star_{\text{cov}} \Phi = \Phi \varphi + \frac{i\hbar}{2} \{\varphi, \Phi\}_{\text{cov}},$$

since φ has tensor degree 1 and hence the higher order contributions in \star_{cov} all vanish. Thus for $\varphi \in \ker F_M$ we get $\Phi \star_{\text{cov}} \varphi = \Phi \varphi = \varphi \star_{\text{cov}} \Phi$. But this shows that

$$\langle \ker F_M \rangle \otimes \mathbb{C} = \mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C}) \star_{\text{cov}} \ker F_M \star_{\text{cov}} \mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C}).$$

Since $\ker F_M$ consists of real sections, it is clear that $\langle \ker F_M \rangle \otimes \mathbb{C}$ is a \star -ideal. \square

6.5 The Relation between the Canonical and the Covariant Poisson Algebra

Let us now relate the two Poisson algebras $\mathbf{S}^\bullet(V_\Sigma)$ and $\mathbf{S}^\bullet(V_{\text{cov}})$. In view of (6.21) it is reasonable to relate a section $\varphi \in \Gamma_0^\infty(E^*)$ to sections $\varphi_0, \dot{\varphi}_0 \in \Gamma_0^\infty(E_\Sigma^*)$ by defining

$$\varphi_0 = \iota^\# \left(\nabla_n^{E^*} F_M(\varphi) \right) \quad \text{and} \quad \dot{\varphi}_0 = -\iota^\#(F_M(\varphi)), \quad (6.33)$$

thereby defining a linear map

$$\varrho_\Sigma: \Gamma_0^\infty(E^*) \ni \varphi \mapsto (\varphi_0, \dot{\varphi}_0) \in \Gamma_0^\infty(E_\Sigma^*) \oplus \Gamma_0^\infty(E_\Sigma^*), \quad (6.34)$$

with $\varphi_0, \dot{\varphi}_0$ given as in (6.33). This map ϱ_Σ has the following property:

Lemma 6.10 *The map ϱ_Σ is continuous and for all solutions $u \in \Gamma_{\text{sc}}^\infty(E)$ of the wave equation with initial conditions u_0, \dot{u}_0 we have*

$$\varphi(u) = \varrho_\Sigma(\varphi)(u_0, \dot{u}_0). \quad (6.35)$$

Proof. The propagator F_M gives a continuous map into $\Gamma_{\text{sc}}^\infty(E^*)$ and the covariant derivative is clearly continuous, too, mapping $\Gamma_{\text{sc}}^\infty(E^*)$ into $\Gamma_{\text{sc}}^\infty(E^*)$. For every compact subset $K \subseteq \Sigma$ the restriction

$$\iota^\#: \Gamma_{J_M(K)}^\infty(E^*) \longrightarrow \Gamma_K^\infty(E_\Sigma^*)$$

is a continuous map between Fréchet spaces. But then also $\iota^\#: \Gamma_{\text{sc}}^\infty(E^*) \longrightarrow \Gamma_0^\infty(E_\Sigma^*)$ is continuous by the universal property of LF topologies. This shows the continuity of ϱ_Σ , the equality in (6.35) is just (6.21). \square

Since $\mathbf{S}^\bullet(V_{\text{cov}})$ is freely generated by V_{cov} we get a unique unital algebra homomorphism extending ϱ_Σ which we still denote by the same symbol

$$\varrho_\Sigma: \mathbf{S}^\bullet(V_{\text{cov}}) \longrightarrow \mathbf{S}^\bullet(V_\Sigma). \quad (6.36)$$

Since (6.34) is continuous, also (6.36) is continuous as linear map from $\mathbf{S}_{R,1}^\bullet(V_{\text{cov}})$ to $\mathbf{S}_{R,1}^\bullet(V_\Sigma)$ by Lemma 5.1. Moreover, we have

$$\Phi(u) = \varrho_\Sigma(\Phi)(u_0, \dot{u}_0) \quad (6.37)$$

for all $\Phi \in \mathbf{S}^\bullet(V_{\text{cov}})$ and all solutions $u \in \Gamma_{\text{sc}}^\infty(E)$ of the wave equation $Du = 0$ with initial conditions (u_0, \dot{u}_0) . This is clear since evaluation of an element in the symmetric algebra on a point is a homomorphism and ϱ_Σ is a homomorphism as well. Since we only have to check the equality of two homomorphism on generators, (6.35) is all we need to conclude (6.37).

Lemma 6.11 *The algebra homomorphism ϱ_Σ is a Poisson morphism as well as a continuous $*$ -algebra homomorphism*

$$\varrho_\Sigma: \mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C}, \star_{\text{cov}}) \longrightarrow \mathcal{W}_R(V_\Sigma \otimes \mathbb{C}, \star_\Sigma). \quad (6.38)$$

Proof. Thanks to Proposition 2.15 and Proposition 5.2 we only have to show that (6.34) is a Poisson map. Thus let $\varphi, \psi \in \Gamma_0^\infty(E^*)$ be given and let $(\varphi_0, \dot{\varphi}_0) = \varrho_\Sigma(\varphi)$ as well as $(\psi_0, \dot{\psi}_0) = \varrho_\Sigma(\psi)$ be their images in V_Σ under ϱ_Σ . Consider now $u = (F_M \psi)^\# = G_M(\psi^\#) \in \Gamma_{\text{sc}}^\infty(E)$ which is a solution of the wave equation $Du = 0$ with initial conditions $u_0 = \iota^\# u = \iota^\#(F_M(\psi))^\#$ and $\dot{u}_0 = \iota^\#(\nabla_n^E u) = \iota^\#(\nabla_n^{E^*} F_M(\psi))^\#$. Here we use that ∇^E is metric and hence compatible with the musical isomorphism $\#$ induced by h . Now we have

$$\begin{aligned} & \Lambda_\Sigma((\varphi_0, \dot{\varphi}_0), (\psi_0, \dot{\psi}_0)) \\ &= - \int_\Sigma \left(\left(\iota^\# \nabla_n^{E^*} F_M(\varphi) \right) \cdot \underbrace{\left(\iota^\# F_M(\psi) \right)^\#}_{u_0} - \left(\iota^\# F_M(\varphi) \right) \cdot \underbrace{\left(\iota^\# \nabla_n^{E^*} F_M(\psi) \right)^\#}_{\dot{u}_0} \right) \mu_\Sigma \\ &\stackrel{(6.21)}{=} - \int_M \varphi \cdot u \, \mu_g \\ &= - \int_M h^{-1}(\varphi, F_M(\psi)) \, \mu_g \\ &= \Lambda_{\text{cov}}(\varphi, \psi). \end{aligned}$$

Clearly, ϱ_Σ is real and hence commutes with the complex conjugation. \square

Lemma 6.12 *The kernel of ϱ_Σ coincides with the Poisson ideal generated by $\ker F_M$ which consists of those elements in $S^\bullet(V_{\text{cov}})$ which vanish on all solutions.*

Proof. Clearly, the kernel of $\varrho_\Sigma|_{V_{\text{cov}}}$ is given by $\ker F_M$ by Lemma 6.8. This implies $\ker \varrho_\Sigma = \langle \ker F_M \rangle$ in general. The second statement then follows from (6.37) at once. \square

This statement has a very natural physical interpretation: if we view $\Phi, \Psi \in S^\bullet(V_{\text{cov}})$ as observables of the field theory, their expectation values for a given field configuration are just the evaluations $\Phi(u), \Psi(u) \in \mathbb{R}$, where $u \in u \in \Gamma_{\text{sc}}^\infty(E)$. But since physically only those $u \in \Gamma_{\text{sc}}^\infty(E)$ occur which also satisfy the wave equation $Du = 0$, we have to identify the observables Φ and Ψ as soon as they coincide on the solutions. This is the case iff $\Phi - \Psi \in \langle \ker F_M \rangle$.

Lemma 6.13 *The homomorphism ϱ_Σ is surjective.*

Proof. Let $\varphi_0, \dot{\varphi}_0 \in \Gamma_0^\infty(E_\Sigma^*)$ be given. Then there is a (unique) solution $\Phi \in \Gamma_{\text{sc}}^\infty(E^*)$ of the wave equation $D^T \Phi = 0$ with the initial conditions

$$\iota^\# \Phi = -\dot{\varphi}_0 \quad \text{and} \quad \iota^\# \nabla_n^{E^*} \Phi = \varphi_0,$$

since D^T is normally hyperbolic as well. By (6.18) for D^T and F_M we know that $\Phi = F_M \varphi$ for some $\varphi \in \Gamma_0^\infty(E^*)$. Then $\varrho_\Sigma(\varphi) = (\varphi_0, \dot{\varphi}_0)$ follows. \square

We can collect now the above results in the following statement leading to the comparison between the covariant and the canonical Poisson bracket and their Weyl algebras, see [25, Thm. 4.4.22] for the classical part:

Theorem 6.14 Fix $R > \frac{1}{2}$. Let (M, g) be a globally hyperbolic spacetime and let $E \rightarrow M$ be a real vector bundle with fiber metric h and metric connection ∇^E . Moreover, let $D = \square^\nabla + B$ with $B = B^* \in \Gamma^\infty(\text{End}(E))$ be a symmetric normally hyperbolic differential operator on E and denote by F_M the propagator of its adjoint D^* . Finally, let $\iota: \Sigma \rightarrow M$ be a smooth spacelike Cauchy surface.

i.) The following subspaces of $S^\bullet(V_{\text{cov}})$ coincide:

- The vanishing ideal if the solutions of the wave function $Du = 0$.
- The Poisson ideal generated by the Casimir elements $\varphi \in V_{\text{cov}}$.
- The ideal $\langle \ker F_M \rangle$.
- The kernel of the Poisson homomorphism $\varrho_\Sigma: S^\bullet(V_{\text{cov}}) \rightarrow S^\bullet(V_\Sigma)$.

ii.) The locally convex quotient Poisson algebra $S_{R,1}^\bullet(V_{\text{cov}})/\langle \ker F_M \rangle$ is canonically isomorphic to the Poisson algebra $S_{R,1}^\bullet(V_{\text{cov}}/\ker F_M)$, with the Poisson bracket coming from (6.31) defined on classes.

iii.) The Poisson homomorphism ϱ_Σ induces a continuous Poisson isomorphism

$$\varrho_\Sigma: S_{R,1}^\bullet(V_{\text{cov}})/\langle \ker F_M \rangle \rightarrow S_{R,1}^\bullet(V_\Sigma). \quad (6.39)$$

iv.) The locally convex quotient $*$ -algebra $\mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C}, \star_{\text{cov}})/(\langle \ker F_M \rangle \otimes \mathbb{C})$ is canonically $*$ -isomorphic to the Weyl algebra $\mathcal{W}_R((V_{\text{cov}}/\ker F_M) \otimes \mathbb{C}, \star_{\text{cov}})$ with \star_{cov} coming from the Poisson bracket (6.31) of $V_{\text{cov}}/\ker F_M$.

v.) The $*$ -homomorphism ϱ_Σ induces a continuous $*$ -isomorphism

$$\varrho_\Sigma: \mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C}, \star_{\text{cov}})/(\langle \ker F_M \rangle \otimes \mathbb{C}) \rightarrow \mathcal{W}_R(V_\Sigma \otimes \mathbb{C}, \star_\Sigma). \quad (6.40)$$

Proof. The only things left to prove are the continuity statements with respect to the quotient topologies. But these follow from the general situation discussed in Lemma 4.5 and Corollary 4.6. \square

Remark 6.15 It follows that the evaluation of $\Phi \in S^\bullet(V_{\text{cov}})$ on a solution $u \in \Gamma_{\text{sc}}^\infty(E)$ of the wave equation depends only on the equivalence class of Φ modulo $\langle \ker F_M \rangle$. Hence this evaluation yields a well-defined and still continuous linear functional on $S_{R,1}^\bullet(V_{\text{cov}})/\langle \ker F_M \rangle$ and on $\mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C}, \star_{\text{cov}})/(\langle \ker F_M \rangle \otimes \mathbb{C})$. This way, the elements in these quotients can be thought of as observables of the field theory described by the wave equation $Du = 0$.

6.6 Locality and Time-Slice Axiom

In this last section we collect some further properties of the covariant Poisson bracket and its Weyl algebra quantization as required by the Haag-Kastler approach to (quantum) field theory [18]: locality and the time-slice axiom.

Let $U \subseteq M$ be a non-empty open subset. Then we denote by $\tilde{\mathcal{A}}_{\text{cl}}(U) \subseteq S^\bullet(V_{\text{cov}})$ the unital Poisson subalgebra generated by those $\varphi \in \Gamma_0^\infty(E^*)$ with $\text{supp } \varphi \subseteq U$. Analogously, we define $\tilde{\mathcal{A}}(U) \subseteq \mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C}, \star_{\text{cov}})$ to be the unital $*$ -subalgebra generated by those $\varphi \in \Gamma_0^\infty(E^*)$ with $\text{supp } \varphi \subseteq U$. For $U = \emptyset$ we set $\tilde{\mathcal{A}}_{\text{cl}}(\emptyset) = \mathbb{C}\mathbb{1} = \tilde{\mathcal{A}}(\emptyset)$. Finally, we set $\mathcal{A}_{\text{cl}}(U)$ and $\mathcal{A}(U)$ for the images of $\tilde{\mathcal{A}}_{\text{cl}}(U)$ and $\tilde{\mathcal{A}}(U)$ in the quotients $S^\bullet(V_{\text{cov}})/\langle \ker F_M \rangle$ and $\mathcal{W}_R(V_{\text{cov}} \otimes \mathbb{C}, \star_{\text{cov}})/(\langle \ker F_M \rangle \otimes \mathbb{C})$, respectively. Clearly, $\mathcal{A}_{\text{cl}}(M)$ and $\mathcal{A}(M)$ yield again everything. Then the following properties are obvious from the causal properties of F_M :

Proposition 6.16 (Local net of observables) Let $U, U' \subseteq M$ be open subsets of M .

i.) We have the locality property

$$\{\mathcal{A}_{\text{cl}}(U), \mathcal{A}_{\text{cl}}(U')\}_{\text{cov}} = 0 \quad \text{and} \quad [\mathcal{A}(U), \mathcal{A}(U')]_{\star_{\text{cov}}} = 0 \quad (6.41)$$

whenever U and U' are spacelike.

ii.) For $U \subseteq U'$ we have

$$\mathcal{A}_{\text{cl}}(U) \subseteq \mathcal{A}_{\text{cl}}(U') \quad \text{and} \quad \mathcal{A}(U) \subseteq \mathcal{A}(U') \quad (6.42)$$

iii.) We have

$$\bigcup_{U \subseteq M_{\text{open}}} \mathcal{A}_{\text{cl}}(U) = \mathcal{A}_{\text{cl}}(M) \quad \text{and} \quad \bigcup_{U \subseteq M_{\text{open}}} \mathcal{A}(U) = \mathcal{A}(M). \quad (6.43)$$

Proof. For $\phi, \psi \in \Gamma_0^\infty(E^*)$ we clearly have $\{\phi, \psi\}_{\text{cov}} = 0 = [\phi, \psi]_{\star_{\text{cov}}}$ whenever U and U' are spacelike and $\text{supp } \phi \subseteq U$ and $\text{supp } \psi \subseteq U'$. This follows immediately from (6.31) and the fact that $\text{supp } F_M(\phi) \subseteq J_M(\text{supp } \phi) \subseteq J_M(U)$ which does not intersect U' . Then the Leibniz rule shows (6.41) in general. The remaining statements are clear. \square

The time-slice axiom requires that a small neighbourhood of a Cauchy surface contains already all the information about the observables. In our framework, this can be formulated as follows:

Proposition 6.17 (Time-slice axiom) *Let $\iota: \Sigma \longrightarrow M \cong \mathbb{R} \times \Sigma$ be a smooth Cauchy surface and let $\epsilon > 0$. Then*

$$\mathcal{A}_{\text{cl}}(\Sigma_\epsilon) = \mathcal{A}_{\text{cl}}(M) \quad \text{and} \quad \mathcal{A}(\Sigma_\epsilon) = \mathcal{A}(M), \quad (6.44)$$

where $\Sigma_\epsilon = (-\epsilon, \epsilon) \times \Sigma$ is the ϵ -time slice around Σ .

Proof. First we note that Σ_ϵ is a globally hyperbolic spacetime by its own and the inclusion $\Sigma_\epsilon \subseteq M$ is causally compatible, i.e. we have $J_{\Sigma_\epsilon}^\pm(p) = J_M^\pm(p) \cap \Sigma_\epsilon$ for all $p \in \Sigma_\epsilon$. Restricting D and D^\top to Σ_ϵ gives globally hyperbolic differential operators with Green operators $G_{\Sigma_\epsilon}^\pm$ and $F_{\Sigma_\epsilon}^\pm$, respectively. By the uniqueness of the Green operators we have for $\varphi \in \Gamma_0^\infty(E^*|_{\Sigma_\epsilon})$ the equality

$$F_{\Sigma_\epsilon}^\pm(\varphi) = F_M^\pm(\varphi)|_{\Sigma_\epsilon}.$$

Thus the covariant Poisson bracket for $\mathbf{S}^\bullet(\Gamma_0^\infty(E^*|_{\Sigma_\epsilon}))$ is the restriction of the one from $\mathbf{S}^\bullet(\Gamma_0^\infty(E^*))$ to the subalgebra $\mathbf{S}^\bullet(\Gamma_0^\infty(E^*|_{\Sigma_\epsilon}))$. Next, let $\varphi \in \Gamma_0^\infty(E^*|_{\Sigma_\epsilon})$ be given. Then in the condition $\varphi(u) = 0$ only $u|_{\Sigma_\epsilon}$ enters. This shows that also the kernels of F_M and F_{Σ_ϵ} correspond, i.e. we have

$$\ker F_{\Sigma_\epsilon} = \ker \left(F_M|_{\Gamma_0^\infty(E^*|_{\Sigma_\epsilon})} \right).$$

Putting this together we conclude that the two isomorphisms ϱ_Σ with respect to the Cauchy surface Σ , once sitting inside M and the other time sitting inside $(-\epsilon, \epsilon) \times \Sigma$, give the desired isomorphism needed for (6.44). \square

For more information on the locality properties and the time-slice axiom in the context of quantum field theories on globally hyperbolic spacetimes we refer to the recent works [8, 19] as well as [1, Chap. 4], where the C^* -algebraic version and the functorial aspects of the above construction are discussed in detail.

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